

# Periodic solutions for nonlinear hyperbolic evolution systems

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## Abstract

We shall deal with the periodic problem for nonlinear perturbations of abstract hyperbolic evolution equations generating an evolution system of contractions. We prove an averaging principle for the translation along trajectories operator associated to the nonlinear evolution system, expressed in terms of the topological degree. The abstract results shall be applied to the damped hyperbolic partial differential equation.

## 1 Introduction

We shall be concerned with  $T$ -periodic solutions of the nonlinear evolution equation

$$(P) \quad \dot{u}(t) = A(t)u(t) + F(t, u(t)), \quad t \in [0, T]$$

where  $T > 0$  is fixed,  $\{A(t)\}_{t \in [0, T]}$  is a family of linear operators on a separable Banach space  $E$  satisfying the so-called hyperbolic conditions and  $F : [0, T] \times E \rightarrow E$  is a  $T$ -periodic in time continuous map satisfying the local Lipschitz condition with respect to the second variable and having sublinear growth, uniformly with respect to time. Moreover, it is also assumed that there is  $\omega > 0$  such that

$$\|S_{A(t)}(s)\| \leq e^{-\omega s} \quad \text{for } t \in [0, T] \text{ and } s \geq 0,$$

where  $S_{A(t)}$  stands for the  $C_0$  semigroup generated by the operator  $A(t)$ , and that there is  $k \in [0, \omega)$  such that

$$\beta(F([0, T] \times Q)) \leq k\beta(Q) \quad \text{for any bounded } Q \subset E,$$

where  $\beta$  denotes the Hausdorff measure of noncompactness. Under these assumptions, the translation along trajectories operators  $\Phi_t : E \rightarrow E$ ,  $t \in [0, T]$ , given by  $\Phi_t(x) := u(t; x)$ ,  $x \in E$ , where  $u(\cdot; x)$  stands for the solution of (P) with the initial condition  $u(0) = x$ , are well-defined and continuous. Moreover, for  $t \in [0, T]$ , one has  $\beta(\Phi_t(Q)) \leq e^{-(\omega-k)t}\beta(Q)$  for any bounded  $Q \subset E$ . This enables us to consider the topological degree of  $I - \Phi_t$  and search  $T$ -periodic solutions corresponding to the fixed points of  $\Phi_T$ . Our approach is based on the averaging idea, which says that if increasing the frequency in (P), i.e. considering equations  $\dot{u}(t) = A(t/\lambda)u(t) + F(t/\lambda, u(t))$  with  $\lambda \rightarrow 0^+$ , then their solutions converge to solutions of the averaged equation

$$\dot{u}(t) = \hat{A}u(t) + \hat{F}(u(t))$$

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**2000 Mathematical Subject Classification:** 47J35, 47J15, 37L05

**Key words:** semigroup, evolution system, evolution equation, topological degree, periodic solution

where  $\widehat{A} + \widehat{F}$  is the time averaged right-hand side of  $(P)$  (the precise meaning is explained in the sequel – see Theorem 4.1). Therefore, after rescaling time, we study  $T$ -periodic solutions of equations

$$(P_\lambda) \quad \dot{u}(t) = \lambda A(t)u(t) + \lambda F(t, u(t)), \quad t \in [0, T]$$

by means of the associated translation along trajectories operator  $\Phi_T^{(\lambda)}$ . We prove that, for small  $\lambda > 0$ , the topological degree of  $I - \Phi_T^{(\lambda)}$ , with respect to a proper open bounded  $U \subset E$ , is equal to the topological degree  $\text{Deg}(\widehat{A} + \widehat{F}, U)$  – see Theorem 4.4. This formula will imply the existence of  $T$ -periodic solutions provided  $\text{Deg}(\widehat{A} + \widehat{F}, U) \neq 0$ . In some natural cases the geometry of the right-hand side allows concluding the nontriviality of the topological degree and get some a priori bounds estimates, which provide effective criteria for the existence of  $T$ -periodic solutions – see Theorem 4.11.

The abstract hyperbolic type linear or semi-linear systems and their applications to partial differential equations were developed by Kato (see e.g. [9]) and Tanabe (see e.g. [17] and references therein). Some existence results for initial value problems associated with nonlinear perturbations of evolution systems are standard and can be found e.g. in [14]. As we need the continuity of translation along trajectories and some related homotopies, we have to verify the continuity and compactness of solutions as functions of initial data and parameters. Moreover, due to some infinitesimal passages related to the averaging method used in the paper, a parameterized version of the representation formula must be derived. As a tool we use the topological degree for so called  $k$ -set contraction vector fields due to Sadovskii (see [1] and references therein) and Nussbaum (see [11]). The topological degree for maps of the form  $A + F$ , where an invertible operator  $A$  generates a  $C_0$  semigroup and  $F$  is a continuous  $k$ -set contraction is obtained as the degree of vector field  $I + A^{-1}F$ , which is a standard – see e.g. [?] and some comments on the specific properties that we use are in [4]. Averaging methods combined with topological degree and fixed point index were used in [7] to find periodic solutions for time dependent vector fields on finite dimensional manifolds. Analogues of this method, in the case of infinite dimensional Banach spaces was stated in [2], [3], where periodic solutions for the equations of the form  $\dot{u}(t) = Au(t) + F(t, u(t))$ , with  $A$  generating compact semigroups, were derived. Also averaging methods together with Rybakowski's version of the Conley index were used in [15], where the existence of so-called recurrent solutions is studied for nonautonomous parabolic equations. Periodic solutions for nonautonomous damped hyperbolic equations has been also thoroughly studied in [12] and [13]. The present paper is a continuation of [4] where the periodic problem is considered in the case where  $A$  generates a  $C_0$ -semigroup of strict contractions and  $F$  is a perturbation, i.e. the situation applicable to damped hyperbolic equations.

The paper is organized as follows. In Section 2, we prove a parameterized version of the representation theorem, which is a useful framework for limit passages concerned with evolution systems at all and also those considered in the next sections. Section 3 is devoted to the properties of the translation along trajectories operator such as the existence, continuity with respect to the parameter and compactness. In Section 4 we deal with the main result of the paper, that is the averaging method for periodic solutions of  $(P)$ . Section 5 provides an example of application to second order hyperbolic partial differential equations.

## 2 General Representation Theorem

We start with a parameterized version of Theorem 3.5 from [14, Ch. 3].

**Theorem 2.1** *Let  $L : (0, +\infty) \times [0, 1] \rightarrow \mathcal{L}(E, E)$ , where  $E$  is a Banach space, be a mapping such that*

$$(1) \quad \|L(\lambda, \mu)\| \leq 1 \quad \text{for } \lambda > 0, \mu \in [0, 1]$$

*and there is a dense subspace  $V$  of  $E$  such that*

$$(2) \quad \lim_{\lambda \rightarrow 0^+, \mu \rightarrow \mu_0} \lambda^{-1}(L(\lambda, \mu)v - v) = A^{(\mu_0)}v \quad \text{for } v \in V, \mu_0 \in [0, 1],$$

*where, for each  $\mu \in [0, 1]$ ,  $A^{(\mu)} : D(A^{(\mu)}) \rightarrow E$  is a linear operator such that  $V \subset D(A^{(\mu)})$  and  $(a_\mu I - A^{(\mu)})V$  is dense in  $E$  for some  $a_\mu > 0$ .*

*Then*

- (i) *for any  $\mu \in [0, 1]$ , the operator  $A^{(\mu)}$  is closable and its closure  $\overline{A^{(\mu)}}$  generates a  $C_0$  semigroup of contractions  $\{S_{\overline{A^{(\mu)}}}(t) : E \rightarrow E\}_{t \geq 0}$ ;*
- (ii) *for any sequence of a positive integers  $(k_n)$  and a sequence  $(\lambda_n)$  in  $(0, +\infty)$  such that  $k_n \rightarrow \infty$ ,  $k_n \lambda_n \rightarrow t$  as  $n \rightarrow +\infty$ , for some  $t \geq 0$ , and any  $(\mu_n)$  in  $[0, 1]$  with  $\mu_n \rightarrow \mu_0$ ,*

$$(3) \quad \lim_{n \rightarrow \infty} L(\lambda_n, \mu_n)^{k_n} x = S_{\overline{A^{(\mu_0)}}}(t)x \quad \text{for each } x \in E;$$

- (iii) *for sequences  $(k_n)$ ,  $(\lambda_n)$ ,  $(\mu_n)$  and  $t \geq 0$  as in (ii)*

$$(4) \quad \lim_{n \rightarrow \infty} \lambda_n(I + L(\lambda_n, \mu_n) + L(\lambda_n, \mu_n)^2 + \dots + L(\lambda_n, \mu_n)^{k_n-1})x \rightarrow \int_0^t S_{\overline{A^{(\mu_0)}}}(\tau)x d\tau$$

*for each  $x \in E$ .*

In the proof we shall use the following two Lemmata.

**Lemma 2.2** (see [14, Ch. 3, Theorem 4.5]) *If  $(A_n)_{n \geq 1}$  is a sequence of operators generating  $C_0$  semigroups  $\{S_{A_n}(t)\}_{t \geq 0}$ ,  $n \geq 1$ , and  $A : V \rightarrow E$  is a linear operator, where  $V$  is a dense subspace of  $E$ , with the following properties*

- (a) *there are  $M \geq 1$  and  $\omega \in \mathbb{R}$  such that  $\|S_{A_n}(t)\| \leq Me^{\omega t}$  for any  $n \geq 1$ ;*
- (b) *for every  $v \in V$ ,  $A_n v \rightarrow Av$  as  $n \rightarrow \infty$ ;*
- (c) *there exists  $\mu_0 > \omega$  such that  $(\mu_0 I - A)V$  is dense in  $E$ ,*

*then the closure  $\overline{A}$  of  $A$  generates a  $C_0$  semigroup  $\{S_{\overline{A}}(t)\}_{t \geq 0}$  such that*

$$\|S_{\overline{A}}(t)\| \leq Me^{\omega t} \quad \text{for } t \geq 0$$

*and*

$$\lim_{n \rightarrow \infty} S_{A_n}(t)x = S_{\overline{A}}(t)x \quad \text{for } t \geq 0, x \in E.$$

*The above convergence is uniform with respect to  $t$  from bounded intervals.*

**Lemma 2.3** (see [14, Ch. 3, Corollary 5.2]) *If  $T \in \mathcal{L}(E, E)$  and  $\|T\| \leq 1$ , then for any integer  $n \geq 0$  and  $x \in E$*

$$\|e^{(T-I)n}x - T^n x\| \leq \sqrt{n}\|x - Tx\|.$$

**Proof of Theorem 2.1.** (i) Define  $A_{\lambda}^{(\mu)} : E \rightarrow E$  by  $A_{\lambda}^{(\mu)} := \lambda^{-1}(L(\lambda, \mu) - I)$  and for any  $\lambda > 0$ ,  $\mu \in [0, 1]$  and  $t \geq 0$ , put  $S_{\lambda}^{(\mu)}(t) := \exp(tA_{\lambda}^{(\mu)})$ . Clearly, in view of (1), for any  $\lambda > 0$  and  $\mu \in [0, 1]$

$$(5) \quad \|S_{\lambda}^{(\mu)}(t)\| \leq e^{-t/\lambda} \sum_{k=0}^{\infty} (t/\lambda)^k \frac{\|L(\lambda, \mu)^k\|}{k!} \leq e^{-t/\lambda} \sum_{k=0}^{\infty} \frac{(t/\lambda)^k}{k!} = 1.$$

If  $\lambda_n \rightarrow 0^+$  and  $\mu_n \rightarrow \mu_0$ , then due to (2),

$$\lim_{n \rightarrow \infty} A_{\lambda_n}^{(\mu_n)} v = A^{(\mu_0)} v \quad \text{for } v \in V.$$

By the assumption, there is  $a_{\mu_0} > 0$  such that  $(a_{\mu_0}I - A^{(\mu_0)})V$  is dense in  $E$  and, in view of Lemma 2.2, we infer that  $A^{(\mu_0)}$  is closable and its closure  $\overline{A^{(\mu_0)}}$  generates  $C_0$  a semigroup  $\{S_{\overline{A^{(\mu_0)}}}(t)\}_{t \geq 0}$  of bounded linear operators on  $E$ , such that  $\|S_{\overline{A^{(\mu_0)}}}(t)\| \leq 1$  for any  $t \geq 0$  and furthermore

$$(6) \quad S_{\lambda_n}^{(\mu_n)}(t)x \rightarrow S_{\overline{A^{(\mu_0)}}}(t)x \quad \text{for any } x \in E, \text{ as } n \rightarrow \infty$$

uniformly for  $t$  from bounded subintervals of  $[0, +\infty)$ .

(ii) Let the sequence of a positive integers  $(k_n)$ , the sequence  $(\lambda_n)$  in  $(0, +\infty)$  and  $(\mu_n)$  in  $[0, 1]$  be such that  $k_n \rightarrow \infty$ ,  $k_n \lambda_n \rightarrow t$  as  $n \rightarrow +\infty$ , for some  $t \geq 0$ , and  $\mu_n \rightarrow \mu_0$  as  $n \rightarrow +\infty$ . Then for any  $v \in V$  and  $n \geq 1$

$$(7) \quad \|L(\lambda_n, \mu_n)^{k_n} v - S_{\overline{A^{(\mu_0)}}}(t)v\| \leq \|L(\lambda_n, \mu_n)^{k_n} v - S_{\lambda_n}^{(\mu_n)}(\lambda_n k_n)v\| \\ + \|S_{\lambda_n}^{(\mu_n)}(\lambda_n k_n)v - S_{\overline{A^{(\mu_0)}}}(\lambda_n k_n)v\| + \|S_{\overline{A^{(\mu_0)}}}(\lambda_n k_n)v - S_{\overline{A^{(\mu_0)}}}(t)v\|.$$

By Lemma 2.3 and (2), for any  $v \in V$

$$(8) \quad \|L(\lambda_n, \mu_n)^{k_n} v - S_{\lambda_n}^{(\mu_n)}(\lambda_n k_n)v\| = \|e^{k_n(L(\lambda_n, \mu_n) - I)}v - L(\lambda_n, \mu_n)^{k_n}v\| \\ \leq \sqrt{k_n}\|v - L(\lambda_n, \mu_n)v\| \\ = \sqrt{\lambda_n} \sqrt{k_n \lambda_n} \|\lambda_n^{-1}(v - L(\lambda_n, \mu_n)v)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Consequently, by (1), (5) and the density of  $V$  in  $E$ , we obtain that

$$(9) \quad \|L(\lambda_n, \mu_n)^{k_n} x - S_{\lambda_n}^{(\mu_n)}(\lambda_n k_n)x\| \rightarrow 0 \quad \text{for each } x \in E, \text{ as } n \rightarrow \infty.$$

Furthermore, in view of the uniform convergence on bounded intervals in (6), one has

$$\|S_{\lambda_n}^{(\mu_n)}(\mu_n k_n)x - S_{\overline{A^{(\mu_0)}}}(\lambda_n k_n)x\| \rightarrow 0 \quad \text{for any } x \in E, \text{ as } n \rightarrow \infty.$$

This, together with (7), (9) and the continuity of the semigroup  $S_{\overline{A^{(\mu_0)}}}$ , gives (3).

(iii) Take any  $v \in V$  and observe that

$$\left\| \lambda_n \sum_{k=0}^{k_n-1} L(\lambda_n, \mu_n)^k v - \int_0^t S_{\overline{A^{(\mu_0)}}}(\tau)v d\tau \right\| \leq I_n^{(1)} + I_n^{(2)} + I_n^{(3)},$$

where

$$\begin{aligned} I_n^{(1)} &:= \left\| \lambda_n \sum_{k=0}^{k_n-1} L(\lambda_n, \mu_n)^k v - \lambda_n \sum_{k=0}^{k_n-1} S_{\lambda_n}^{(\mu_n)}(k\lambda_n)v \right\|, \\ I_n^{(2)} &:= \left\| \lambda_n \sum_{k=0}^{k_n-1} S_{\lambda_n}^{(\mu_n)}(k\lambda_n)v - \lambda_n \sum_{k=0}^{k_n-1} S_{A(\mu_0)}(k\lambda_n)v \right\|, \\ I_n^{(3)} &:= \left\| \lambda_n \sum_{k=0}^{k_n-1} S_{A(\mu_0)}(k\lambda_n)v - \int_0^t S_{A(\mu_0)}(\tau)v d\tau \right\|. \end{aligned}$$

First, in view of Lemma 2.3 and (2), one has

$$\begin{aligned} (10) \quad I_n^{(1)} &\leq k_n \lambda_n \max\{\|L(\lambda_n, \mu_n)^k - e^{k(L(\lambda_n, \mu_n) - I)}\| \mid k = 1, \dots, k_n - 1\} \\ &\leq k_n \lambda_n \max\{\sqrt{k}\|v - L(\lambda_n, \mu_n)v\| \mid k = 1, \dots, k_n - 1\} \\ &\leq \sqrt{\lambda_n} (k_n \lambda_n)^{3/2} \|\lambda_n^{-1}(v - L(\lambda_n, \mu_n)v)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Furthermore, by the uniform convergence in (6) on the interval  $[0, \bar{t}]$ , where  $\bar{t} := \sup_{n \geq 1} k_n \lambda_n$ , we get

$$(11) \quad I_n^{(2)} \leq k_n \lambda_n \max\{\|S_{\lambda_n}^{(\mu_n)}(k\lambda_n)v - S_{A(\mu_0)}(k\lambda_n)v\| \mid k = 0, \dots, k_n - 1\} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

It is also clear that  $I_n^{(3)} \rightarrow 0$  as  $n \rightarrow \infty$ , which along with (10) and (11) implies that (4) is satisfied for  $x \in V$ . Finally, since for any  $n \geq 1$

$$\left\| \lambda_n \sum_{k=0}^{k_n-1} L(\lambda_n, \mu_n)^k - \int_0^t S_{A(\mu_0)}(\tau) d\tau \right\| \leq k_n \lambda_n + t < C$$

for some constant  $C > 0$  independent of  $n$  and since  $V$  is dense in  $E$ , one has the required convergence for each  $x \in E$ .  $\square$

### 3 Continuity and compactness properties for solution operator

A family  $\{R(t, s)\}_{0 \leq s \leq t \leq T}$ ,  $T > 0$  of bounded linear operators on a Banach space  $E$  is called an *evolution system* provided  $R(t, t) = I$  for each  $t \in [0, T]$ ,  $R(t, s) = R(t, r)R(r, s)$  if only  $0 \leq s \leq t \leq T$  and for any  $x \in E$ , the map  $(t, s) \mapsto R(t, s)x$  is continuous. A family  $\{R^{(\lambda)}\}_{\lambda \in [0, 1]}$  of evolution systems is called *continuous* if, for any  $x \in E$  and  $(\lambda_n)$  in  $[0, 1]$  with  $\lambda_n \rightarrow \lambda$ ,  $R^{(\lambda_n)}(t, s)x \rightarrow R^{(\lambda)}(t, s)x$  uniformly with respect to  $t, s \in [0, T]$  with  $s \leq t$ .

Evolution systems are naturally determined by time-dependent families of linear operators. Namely, if  $\{A(t)\}_{t \in [0, T]}$  is a family of linear operators on a Banach space  $E$  such that for any  $s \in [0, T]$  and  $x \in E$ , the problem

$$\begin{cases} \dot{u}(t) = A(t)u(t), & t \in [s, T] \\ u(s) = x \end{cases}$$

admits (in some sense) a unique solution  $u_{s,x} : [s, T] \rightarrow E$ , then the corresponding evolution system  $\{R(t, s)\}_{0 \leq s \leq t \leq T}$  is given by  $R(t, s)x := u_{s,x}(t)$ , for  $t \in [s, T]$ . A particular

type of evolution systems – the so-called *hyperbolic evolution systems*, will be discussed in details at the end of this section.

**Proposition 3.1** *Suppose that  $\{R^{(\lambda)}\}_{\lambda \in [0,1]}$  is a continuous family of evolution systems and the operator  $\Sigma : E \times L^1([0, T], E) \times [0, 1] \rightarrow C([0, T], E)$  is given by*

$$\Sigma(x, w, \lambda)(t) := R^{(\lambda)}(t, 0)x + \int_0^t R^{(\lambda)}(t, s)w(s) ds.$$

*Then*

- (i)  $\Sigma$  is continuous;
- (ii) if  $K \subset E$  is relatively compact and  $W \subset L^1([0, T], E)$  is such that there is  $c \in L^1([0, T])$  with  $\|w(t)\| \leq c(t)$  for any  $w \in W$  and a.e.  $t \in [0, T]$ , then  $\Sigma(K \times W \times [0, 1])$  is relatively compact if and only if the set  $\{u(t) \mid u \in \Sigma(K \times W \times [0, 1])\}$  is relatively compact for any  $t \in [0, T]$ .

**Remark 3.2** (a) Under the above notation, if  $t, t+h \in [0, T]$  with  $h > 0$ , then

$$\Sigma(x, w, \lambda)(t+h) = R^{(\lambda)}(t+h, t)\Sigma(x, w, \lambda)(t) + \int_t^{t+h} R^{(\lambda)}(t+h, s)w(s) ds,$$

which follows directly from the definition of  $\Sigma$  and the properties of evolution systems.

(b) If  $\{R^{(\lambda)}\}_{\lambda \in [0,1]}$  is a continuous family of evolution systems, then for any  $x \in E$  the set  $\{R^{(\lambda)}(t, s)x \mid 0 \leq s \leq t \leq T, \lambda \in [0, 1]\}$  is bounded. Hence, in view of the uniform boundedness principle, there exists  $M \geq 0$  such that

$$\|R^{(\lambda)}(t, s)\| \leq M \quad \text{for any } t, s \in [0, T] \text{ with } s \leq t \text{ and } \lambda \in [0, 1].$$

**Proof of Proposition 3.1.** (i) Let  $x_n \rightarrow x_0$  in  $E$ ,  $w_n \rightarrow w_0$  in  $L^1([0, T], E)$  and  $\lambda_n \rightarrow \lambda_0$ . Clearly, by Remark 3.2 one has

$$\begin{aligned} \|R^{(\lambda_n)}(t, 0)x_n - R^{(\lambda_0)}(t, 0)x_0\| &\leq \|R^{(\lambda_n)}(t, 0)x_n - R^{(\lambda_n)}(t, 0)x_0\| + \\ (12) \quad &\quad + \|R^{(\lambda_n)}(t, 0)x_0 - R^{(\lambda_0)}(t, 0)x_0\| \\ &\leq M\|x_n - x_0\| + \|R^{(\lambda_n)}(t, 0)x_0 - R^{(\lambda_0)}(t, 0)x_0\| \end{aligned}$$

and hence, by the continuity of the family  $\{R^{(\lambda)}\}_{\lambda \in [0,1]}$ , we infer that  $\|R^{(\lambda_n)}(t, 0)x_n - R^{(\lambda_0)}(t, 0)x_0\| \rightarrow 0$  as  $n \rightarrow +\infty$ , uniformly with respect to  $t \in [0, T]$ . In a similar manner

$$\begin{aligned} &\left\| \int_0^t R^{(\lambda_n)}(t, s)w_n(s) ds - \int_0^t R^{(\lambda_0)}(t, s)w_0(s) ds \right\| \\ (13) \quad &\leq \int_0^t \|R^{(\lambda_n)}(t, s)w_n(s) - R^{(\lambda_0)}(t, s)w_0(s)\| ds \\ &\leq \int_0^t \|R^{(\lambda_n)}(t, s)w_n(s) - R^{(\lambda_n)}(t, s)w_0(s)\| ds \\ &\quad + \int_0^t \|R^{(\lambda_n)}(t, s)w_0(s) - R^{(\lambda_0)}(t, s)w_0(s)\| ds \\ &\leq M\|w_n - w_0\|_{L^1([0, T], E)} + \int_0^T \varphi_n(s) ds, \end{aligned}$$

where functions  $\varphi_n : [0, T] \rightarrow \mathbb{R}$ ,  $n \geq 1$ , are given by

$$\varphi_n(s) := \sup_{\sigma, \tau \in [0, T], \tau \geq \sigma} \| [R^{(\lambda_n)}(\tau, \sigma) - R^{(\lambda_0)}(\tau, \sigma)] w_0(s) \|.$$

It is easy to check that functions  $\varphi_n$ ,  $n \geq 1$  are measurable and, by the continuity of  $\{R^{(\lambda)}\}_{\lambda \in [0, 1]}$  and Remark 3.2, we infer that, for a.e.  $s \in [0, T]$ ,  $\varphi_n(s) \rightarrow 0$  as  $n \rightarrow +\infty$ . On the other hand  $0 \leq \varphi_n(s) \leq 2M\|w_0(s)\|$ , for  $s \in [0, T]$ , which in view of the Lebesgue dominated convergence theorem gives  $\int_0^T \varphi_n(s) ds \rightarrow 0$  as  $n \rightarrow +\infty$  and together with (12) proves (i).

(ii) Suppose that the set  $\{u(t) \mid u \in \Sigma(K \times W \times [0, 1])\}$  is relatively compact for any  $t \in [0, T]$ . Take any  $\varepsilon > 0$  and fix  $t \in [0, T]$ . Let  $\delta > 0$  be such that  $\int_{[t-\delta, t+\delta] \cap [0, T]} c(s) ds < \varepsilon/3M$ . Suppose that  $t \in [0, T]$ . Since the set  $Q_t := \overline{\{\Sigma(x, w, \lambda)(t) \mid x \in K, w \in W, \lambda \in [0, 1]\}}$  is compact, one may eventually decrease  $\delta > 0$  so that

$$\|R^{(\lambda)}(t+h, t)z - z\| < \varepsilon/2, \quad \text{if } t+h \leq T, h \in [0, \delta), \lambda \in [0, 1], z \in Q_t.$$

Now take  $h \in [0, \delta)$  such that  $t+h \in [0, T]$ . Then, denoting  $\Sigma := \Sigma(x, w, \lambda)$  for any  $(x, w, \lambda) \in K \times W \times [0, 1]$ , one has

$$\begin{aligned} \|\Sigma(t+h) - \Sigma(t)\| &\leq \|\Sigma(t+h) - R^{(\lambda)}(t+h, t)\Sigma(t)\| + \|R^{(\lambda)}(t+h, t)\Sigma(t) - \Sigma(t)\| \\ &\leq \int_t^{t+h} \|R^{(\lambda)}(t+h, s)w(s)\| ds + \|R^{(\lambda)}(t+h, t)\Sigma(t) - \Sigma(t)\| \\ &\leq M \int_t^{t+h} c(s) ds + \varepsilon/2 < \varepsilon. \end{aligned}$$

If  $t \in (0, T]$ , then take any  $\delta_1 \in (0, \min\{t, \delta\})$ . Since the set

$$Q_{t-\delta_1} := \overline{\{\Sigma(x, w, \lambda)(t-\delta_1) \mid x \in K, w \in W, \lambda \in [0, 1]\}}$$

is compact, there exists  $\delta' \in (0, \delta_1]$  such that

$$\|R^{(\lambda)}(t-h, t-\delta_1)z - R^{(\lambda)}(t, t-\delta_1)z\| \leq \varepsilon/3 \quad \text{for any } h \in [0, \delta'), \lambda \in [0, 1], z \in Q_{t-\delta_1}.$$

In consequence, for any  $x \in K$ ,  $w \in W$ ,  $\lambda \in [0, 1]$  and  $h \in [0, \delta')$

$$\begin{aligned} \|\Sigma(t-h) - \Sigma(t)\| &\leq \|\Sigma(t-h) - R^{(\lambda)}(t-h, t-\delta_1)\Sigma(t-\delta_1)\| \\ &\quad + \|R^{(\lambda)}(t-h, t-\delta_1)\Sigma(t-\delta_1) - R^{(\lambda)}(t, t-\delta_1)\Sigma(t-\delta_1)\| \\ &\quad + \|R^{(\lambda)}(t, t-\delta_1)\Sigma(t-\delta_1) - \Sigma(t)\| \\ &\leq \left\| \int_{t-\delta_1}^{t-h} R^{(\lambda)}(t-h, s)w(s) ds \right\| + \varepsilon/3 + \left\| \int_{t-\delta_1}^t R^{(\lambda)}(t, s)w(s) ds \right\| \\ &\leq M \int_{t-\delta_1}^{t-h} c(s) ds + \varepsilon/3 + M \int_{t-\delta_1}^t c(s) ds \leq \varepsilon. \end{aligned}$$

Hence, the set  $\{\Sigma(x, w, \lambda)\}_{x \in K, w \in W, \lambda \in [0, 1]}$  is equicontinuous at any  $t \in [0, T]$ , which due to the Ascoli-Arzelà theorem, completes the proof of (ii).  $\square$

We state basic continuity and compactness results for the translation along trajectories operator for a perturbed (possibly nonlinear) equation

$$(14) \quad \begin{cases} \dot{u}(t) = A(t)u(t) + F(t, u(t)), & t \in [t_0, T] \\ u(t_0) = x_0 \end{cases}$$

where  $\{A(t)\}_{t \in [0, T]}$  is a family of linear operators on Banach space  $E$  with the associated evolution system  $\{R(t, s)\}_{0 \leq s \leq t \leq T}$ ,  $F : [0, T] \times E \rightarrow E$  is a continuous map,  $t_0 \in [0, T]$  and  $x_0 \in E$ . Recall (after [14]) that a continuous function  $u : [t_0, T] \rightarrow E$  is called a *mild solution* of (14) if and only if

$$u(t) = R(t, t_0)x_0 + \int_{t_0}^t R(t, s)F(s, u(s)) ds \quad \text{for any } t \in [t_0, T].$$

For the purpose of next sections we consider below a parameterized framework.

**Proposition 3.3** *For each  $\lambda \in [0, 1]$ , let  $\{A^{(\lambda)}(t)\}_{t \in [0, T]}$  be family of operators on a separable Banach space  $E$  having associated evolution system  $R^{(\lambda)}$ . If the family  $\{R^{(\lambda)}\}_{\lambda \in [0, 1]}$  is continuous and  $F : [0, T] \times E \times [0, 1] \rightarrow E$  is a continuous map being*

*(F<sub>1</sub>)<sub>par</sub> locally Lipschitz in the second variable uniformly with respect to the other variables, i.e. for each  $x \in E$ , there exists  $r_x > 0$  and  $L_x > 0$  such that for each  $t \in [0, T]$ ,  $x_1, x_2 \in B(x, r_x)$  and  $\lambda \in [0, 1]$*

$$\|F(t, x_1, \lambda) - F(t, x_2, \lambda)\| \leq L_x \|x_1 - x_2\|;$$

*(F<sub>2</sub>)<sub>par</sub> of sublinear growth in the second variable uniformly with respect to the others, i.e. there is  $c > 0$  such that*

$$\|F(t, x, \lambda)\| \leq c(1 + \|x\|) \quad \text{for any } x \in E, t \in [0, T], \lambda \in [0, 1];$$

*(F<sub>3</sub>)<sub>par</sub> a  $k$ -set contraction, i.e. there exists  $k \geq 0$  such that*

$$\beta(F([0, T] \times Q \times [0, 1])) \leq k\beta(Q) \quad \text{for any bounded } Q \subset E,$$

*then*

(i) (Existence) *for any  $x \in E$  and  $\lambda \in [0, 1]$ , the initial value problem*

$$(15) \quad \begin{cases} \dot{u}(t) = A^{(\lambda)}(t)u(t) + F(t, u(t), \lambda), & t \in [0, T] \\ u(0) = x \end{cases}$$

*admits a unique mild solution  $u(\cdot; 0, T, x, \lambda)$ ;*

(ii) (Continuity) *if  $(x_n, \lambda_n) \rightarrow (x_0, \lambda_0)$  in  $E \times [0, 1]$ , then*

$$u(\cdot; 0, T, x_n, \lambda_n) \rightarrow u(\cdot; 0, T, x_0, \lambda_0) \quad \text{in } C([0, T], E);$$

(iii) (Compactness) *if additionally, there is  $\omega > 0$  such that*

$$(16) \quad \|R^{(\lambda)}(t, s)\| \leq e^{-\omega(t-s)} \quad \text{for } 0 \leq s \leq t \leq T$$

*and, for any  $t \in [0, T]$ ,  $\Phi_t : E \times [0, 1] \rightarrow E$  is given by  $\Phi_t(x, \lambda) := u(t; 0, T, x, \lambda)$ , then, for any bounded  $Q \subset E$  and  $t \in [0, T]$ , the set  $\Phi_t(Q \times [0, 1])$  is bounded and*

$$\beta(\Phi_t(Q \times [0, 1])) \leq e^{(k-\omega)t}\beta(Q).$$



**Lemma 3.4** (see [6], [8]) *Suppose that  $E$  is a separable Banach space,  $W \subset L^1([a, b], E)$  is countable and there is  $c \in L^1([a, b])$  such that  $\|w(t)\| \leq c(t)$ , for all  $w \in W$  and a.e.  $t \in [a, b]$ , and let  $\phi : [a, b] \rightarrow \mathbb{R}$  be given by  $\phi(t) := \beta(\{w(t) \mid w \in W\})$ . Then  $\phi \in L^1([a, b])$  and*

$$\beta \left( \left\{ \int_a^b w(\tau) d\tau \mid w \in W \right\} \right) \leq \int_a^b \phi(\tau) d\tau.$$

**Lemma 3.5** *Let  $\{R^\lambda\}_{\lambda \in [0,1]}$  be a continuous family of evolution system such that*

$$(17) \quad \|R^{(\lambda)}(t, s)\| \leq Me^{\omega(t-s)} \quad \text{for } 0 \leq s \leq t \leq T \text{ and } \lambda \in [0, 1]$$

where  $M > 0$  and  $\omega \in \mathbb{R}$  are constants. Then, for any bounded  $Q \subset E$  and  $s, t \in [0, T]$  with  $s \leq t$ ,

$$\beta \left( \{R^{(\lambda)}(t, s)x \mid x \in Q, \lambda \in [0, 1]\} \right) \leq Me^{\omega(t-s)} \beta(Q).$$

The proof is analogical to the proof of Lemma 2.1 from [4].

**Proof of Proposition 3.3.** (i) follows by standard arguments (see e.g. [14] and [5]).

The proofs of (ii) and (iii) go in analogy to that of Proposition 3 in [4]. (ii) corresponds also to results from [5] (here we need a version with locally Lipschitz nonlinearity). To see (ii), observe that if  $x_n \rightarrow x_0$  in  $E$  and  $\lambda_n \rightarrow \lambda_0$ , then putting  $u_n(s) := \Phi_s(x_n, \lambda_n)$  for  $s \in [0, T]$ , one has

$$\beta(\{u_n(t)\}_{n \geq 1}) \leq \beta(\{R^{(\lambda_n)}(t, 0)x_n\}_{n \geq 1}) + \beta \left( \left\{ \int_0^t h_{t,n}(s) ds \mid n \geq 1 \right\} \right)$$

where  $h_{t,n}(s) := R^{(\lambda_n)}(t, s)F(s, u_n(s), \lambda_n)$  for  $s \in [0, t]$ . In view of the  $(F_2)_{par}$  and the Gronwall inequality, there is  $M_0 \geq 0$  such that  $\|u_n(s)\| \leq M_0$ , for all  $s \in [0, T]$  and  $n \geq 1$ , which again by  $(F_2)_{par}$  gives  $M_1 \geq 0$  such that  $\|h_{t,n}(s)\| \leq M_1$ , for  $s \in [0, T]$  and  $n \geq 1$ . This allows applying Remark 3.2 and Lemmata 3.4 and 3.5 and as a result, for any  $t \in [0, T]$ ,

$$\begin{aligned} \beta(\{u_n(t)\}_{n \geq 1}) &\leq \beta \left( \left\{ \int_0^t h_{t,n}(s) ds \mid n \geq 1 \right\} \right) \\ &\leq \int_0^t \beta(\{h_{t,n}(s) \mid n \geq 1\}) ds \leq kM \int_0^t \beta(\{u_n(s)\}_{n \geq 1}) \end{aligned}$$

By use of the Gronwall inequality, one gets  $\beta(\{u_n(t)\}_{n \geq 1}) = 0$  for any  $t \in [0, T]$ . Hence, due to the fact that  $u_n = \Sigma(x_n, w_n, \lambda_n)$  with  $w_n(s) := F(s, u_n(s), \lambda_n)$  and Proposition 3.1 (ii) any subsequence of  $(u_n)$  contains a subsequence converging to some  $u_0$ . By Proposition 3.1 (i),  $u_0 = \Sigma(x_0, w_0, \lambda_0)$  with  $w_0(s) := F(s, u_0(s), \lambda_0)$  and therefore  $u_0$  is a unique mild solution of (15). This completes the proof of (ii).

(iii) Let  $Q$  be an arbitrary bounded subset of  $E$  and let  $Q_0$  be a countable subset of  $Q$  such that  $\overline{Q_0} \supset Q$  and  $\Lambda_0$  a countable dense subset of  $[0, 1]$ . First observe that using  $(F_2)_{par}$  and the Gronwall inequality as before, we infer that the sets  $\Phi_t(\overline{Q_0} \times [0, 1])$ ,

$t \in [0, T]$ , are contained in a ball, and clearly, by use of Lemmata 3.5 and 3.4,

$$\begin{aligned} \beta(\Phi_t(Q_0 \times \Lambda_0)) &\leq \beta\left(\{R^{(\lambda)}(t, 0)x \mid x \in Q_0, \lambda \in \Lambda_0\}\right) + \\ &\quad + \beta\left(\left\{\int_0^t w_{x,\lambda,t}(s) ds \mid x \in Q_0, \lambda \in \Lambda_0\right\}\right) \\ &\leq e^{-\omega t} \beta(Q_0) + \int_0^t \beta(\{w_{x,\lambda,t}(s) \mid x \in Q_0, \lambda \in \Lambda_0\}) ds, \end{aligned}$$

where  $w_{x,\lambda,t}(s) := R^{(\lambda)}(t, s)F(s, \Phi_s(x, \lambda), \lambda)$  for  $s \in [0, t]$ . Further observe that, by (16) and Lemma 3.5, for any  $s \in [0, t]$ , one has

$$\begin{aligned} \beta(\{w_{x,\lambda,t}(s) \mid x \in Q_0, \lambda \in \Lambda_0\}) &\leq \beta(\{R^{(\lambda)}(t, s)z \mid z \in F([0, T] \times \Phi_s(Q_0 \times \Lambda_0) \times \Lambda_0), \lambda \in \Lambda_0\}) \\ &\leq e^{-\omega(t-s)} \beta(F([0, T] \times \Phi_s(Q_0 \times \Lambda_0) \times \Lambda_0)) \\ &\leq ke^{-\omega(t-s)} \beta(\Phi_s(Q_0 \times \Lambda_0)). \end{aligned}$$

Combining the previous two inequalities together and applying the Gronwall inequality give  $\beta(\Phi_t(Q_0 \times \Lambda_0)) \leq e^{(k-\omega)t} \beta(Q_0)$  and finally  $\beta(\Phi_t(Q \times [0, 1])) \leq \beta(\Phi_t(\overline{Q_0} \times \overline{\Lambda_0})) \leq \beta(\Phi_t(Q_0 \times \Lambda_0)) = \beta(\Phi_t(Q_0 \times \Lambda_0)) \leq e^{(k-\omega)t} \beta(Q_0) \leq e^{(k-\omega)t} \beta(Q)$ .  $\square$

Now we pass to the hyperbolic case. We shall assume in the rest of this section that  $V$  is a Banach space which is densely and continuously embedded into  $E$ . Given a linear operator  $A : D(A) \rightarrow E$  generating a  $C_0$  semigroup  $\{S_A(t)\}_{t \geq 0}$  of bounded linear operators on  $E$ ,  $V$  is said to be  $A$ -admissible provided  $V$  is an invariant subspace for each  $S_A(t)$  for  $t \geq 0$  and the family of restrictions  $\{S_A(t)_V : V \rightarrow V\}_{t \geq 0}$  ( $S_A(t)_V x := S_A(t)x$ ,  $x \in V$ ) is a  $C_0$  semigroup on  $V$ . Define the *part of  $A$  in the space  $V$*  as a linear operator  $A_V : D(A_V) \rightarrow V$  given by  $D(A_V) := \{v \in D(A) \cap V \mid Av \in V\}$ ,  $A_V v := Av$  for  $v \in D(A_V)$ . In view of [14, Ch. 4, Theorem 5.5], if  $V$  is  $A$ -admissible then  $A_V$  is the generator of the  $C_0$  semigroup  $\{S_A(t)_V\}_{t \geq 0}$ .

**Proposition 3.6** (see [14, Ch. 5, Theorem 3.1]) *Let  $\{A(t)\}_{t \in [0, T]}$  be a family of linear operators on a Banach space  $E$  satisfying the following conditions*

(Hyp<sub>1</sub>)  $\{A(t)\}_{t \in [0, T]}$  *is a stable family of infinitesimal generators of  $C_0$  semigroups, i.e. there are  $M \geq 1$  and  $\omega \in \mathbb{R}$  such that*

$$\|S_{A(t_1)}(s_1) \dots S_{A(t_n)}(s_n)\|_{\mathcal{L}(E, E)} \leq Me^{\omega(s_1 + \dots + s_n)},$$

*whenever  $0 \leq t_1 \leq \dots \leq t_n \leq T$  and  $s_1, \dots, s_n \geq 0$ , where  $\{S_{A(t)}(s)\}_{s \geq 0}$  is the  $C_0$  semigroup generated by  $A(t)$ ;*

(Hyp<sub>2</sub>)  $V$  *is  $A(t)$ -admissible for each  $t \in [0, T]$  and the family  $\{A_V(t)\}_{t \in [0, T]}$  is a stable family of generators of  $C_0$  semigroups with constants  $M_V \geq 1$  and  $\omega_V \in \mathbb{R}$ ;*

(Hyp<sub>3</sub>)  $V \subset D(A(t))$  *and  $A(t) \in \mathcal{L}(V, E)$  for  $t \in [0, T]$  and the mapping  $[0, T] \ni t \mapsto A(t) \in \mathcal{L}(V, E)$  is continuous.*

*Then there exists a unique evolution system  $\{R(t, s)\}_{0 \leq s \leq t \leq T}$  in  $E$  with the following properties*

- (i)  $\|R(t, s)\| \leq Me^{\omega(t-s)}$  for  $0 \leq s \leq t \leq T$ ;
- (ii)  $\frac{\partial^+}{\partial t} R(t, s)v \Big|_{t=s} = A(s)v$  for  $v \in V$ ,  $s \in [0, T]$ ;

$$(iii) \quad \frac{\partial}{\partial s} R(t, s)v = -R(t, s)A(s)v \quad \text{for } v \in V, 0 \leq s \leq t \leq T.$$

Using homotopy invariants will require the continuity of linear evolution systems with respect to parameters.

**Proposition 3.7** *Let, for each  $\lambda \in [0, 1]$ , a family  $\{A^{(\lambda)}(t)\}_{0 \leq t \leq T}$  satisfy conditions (Hyp<sub>1</sub>)–(Hyp<sub>3</sub>) with constants  $M, M_V, \omega, \omega_V$  independent of  $\lambda$  and let  $R^{(\lambda)} = \{R^{(\lambda)}(t, s)\}_{0 \leq s \leq t \leq T}$  be the corresponding evolution systems in  $E$  determined by Proposition 3.6. If, for any  $\lambda_0 \in [0, 1]$ ,*

$$(18) \quad \int_0^T \|A^{(\lambda)}(\tau) - A^{(\lambda_0)}(\tau)\|_{\mathcal{L}(V, E)} d\tau \rightarrow 0 \quad \text{as } \lambda \rightarrow \lambda_0,$$

*then  $\{R^{(\lambda)}\}_{\lambda \in [0, 1]}$  is a continuous family of evolution systems in  $E$ .*

**Proof.** We use the construction from [14, Ch. 5, Theorem 3.1]. Recall that for any  $\lambda \in [0, 1]$  and  $x \in E$

$$R^{(\lambda)}(t, s)x := \lim_{n \rightarrow +\infty} R_n^{(\lambda)}(t, s)x \quad \text{for } 0 \leq s \leq t \leq T,$$

where, for each  $n \geq 1$ , the operator  $R_n^{(\lambda)}(t, s) : E \rightarrow E$  is given by <sup>(2)</sup>

$$R_n^{(\lambda)}(t, s) := \begin{cases} S_j^{(\lambda)}(t-s) & \text{if } s, t \in [t_j^n, t_{j+1}^n], s \leq t, \\ S_k^{(\lambda)}(t-t_k^n) \left( \prod_{j=l+1}^{k-1} S_j^{(\lambda)}(T/n) \right) S_l^{(\lambda)}(t_{l+1}^n - s) & \text{if } s \in [t_l^n, t_{l+1}^n], t \in [t_k^n, t_{k+1}^n], \\ & \text{and } k > l \geq 0, \end{cases}$$

with  $t_j^n := (j/n)T$ ,  $S_j := S_{A^{(\lambda)}(t_j^n)}$ , for  $j = 0, 1, \dots, n$ . Moreover recall (after [14]) that  $\{R_n^{(\lambda)}(t, s)\}_{0 \leq s \leq t \leq T}$  are evolution systems such that

$$(19) \quad \|R_n^{(\lambda)}(t, s)\|_{\mathcal{L}(E, E)} \leq M e^{\omega(t-s)}, \quad \|R_n^{(\lambda)}(t, s)\|_{\mathcal{L}(V, V)} \leq M_V e^{\omega_V(t-s)}, \quad R_n^{(\lambda)}(t, s)V \subset V,$$

for  $0 \leq s \leq t \leq T$  and for any  $v \in V$

$$(20) \quad \frac{\partial}{\partial t} R_n^{(\lambda)}(t, s)v = A_n^{(\lambda)}(t)R_n^{(\lambda)}(t, s)v \quad \text{for } t \notin \{t_0^n, t_1^n, \dots, t_n^n\}, s \leq t,$$

$$(21) \quad \frac{\partial}{\partial s} R_n^{(\lambda)}(t, s)v = -R_n^{(\lambda)}(t, s)A_n^{(\lambda)}(s)v \quad \text{for } s \notin \{t_0^n, t_1^n, \dots, t_n^n\}, s \leq t,$$

with  $A_n^{(\lambda)}(t) := A^{(\lambda)}(t_k^n)$  if  $t_k^n \leq t < t_{k+1}^n$  for  $k = 0, \dots, n-1$  and  $A_n^{(\lambda)}(T) := A^{(\lambda)}(T)$ .

Observe that in view of (Hyp<sub>3</sub>) for any  $\lambda \in [0, 1]$ , one has  $\|A_n^{(\lambda)}(t) - A^{(\lambda)}(t)\|_{\mathcal{L}(V, E)} \rightarrow 0$  as  $n \rightarrow +\infty$  uniformly with respect to  $t \in [0, T]$ . Fix any  $v \in V$ ,  $\lambda, \mu \in [0, 1]$ ,  $n \geq 1$  and  $s, t \in [0, T]$  with  $s < t$  and define  $\phi : [s, t] \rightarrow E$  by  $\phi(r) := R_n^{(\lambda)}(t, r)R_n^{(\mu)}(r, s)v$ . In view of (19), (20) and (21) the map  $\phi$  is differentiable on  $[s, t]$  except finite number of points and

$$\begin{aligned} R_n^{(\mu)}(t, s)v - R_n^{(\lambda)}(t, s)v &= \phi(t) - \phi(s) = \int_s^t \phi'(r) dr, \\ &= \int_s^t \left( R_n^{(\lambda)}(t, r)(A_n^{(\mu)}(r)) - A_n^{(\lambda)}(r)R_n^{(\mu)}(r, s)v \right) dr. \end{aligned}$$

---

<sup>2</sup>Here we adopt the convention that  $\prod_{k=1}^n T_k := T_n \circ T_{n-1} \circ \dots \circ T_1$ , for the sequence  $T_1, T_2, \dots, T_n$  of bounded operators on  $E$ .

Hence, by (19),

$$\|R_n^{(\mu)}(t, s)v - R_n^{(\lambda)}(t, s)v\| \leq MM_V e^{(\omega + \omega_V)T} \|v\|_V \int_0^T \|A_n^{(\mu)}(r) - A_n^{(\lambda)}(r)\|_{\mathcal{L}(V, E)} dr.$$

Passing to the limit with  $n \rightarrow +\infty$ , we get

$$\|R^{(\mu)}(t, s)v - R^{(\lambda)}(t, s)v\| \leq MM_V e^{(\omega + \omega_V)T} \|v\|_V \int_0^T \|A^{(\mu)}(r) - A^{(\lambda)}(r)\|_{\mathcal{L}(V, E)} dr$$

and in consequence,  $R^{(\lambda)}(t, s)v \rightarrow R^{(\lambda_0)}(t, s)v$  for any  $v \in V$ , as  $\lambda \rightarrow \lambda_0$  and the convergence is uniform with respect to  $s, t$ . Using the density of  $V$  in  $E$ , we complete the proof since  $\|R^{(\lambda)}(t, s)\| \leq M e^{\omega(t-s)}$  for  $\lambda \in [0, 1]$  and  $0 \leq s \leq t \leq T$ .  $\square$

The following criterion for verification of conditions  $(Hyp_1) - (Hyp_3)$  is useful in applications.

**Proposition 3.8** ([14, Ch. 5, Theorem 4.8]) *Suppose that a family  $\{A(t)\}_{t \in [0, T]}$ , where  $D(A(t)) = D$  for any  $t \in [0, T]$  and some  $D \subset E$ , is stable and for each  $v \in D$  the mapping  $[0, T] \ni t \mapsto A(t)v \in E$  is continuously differentiable. Then, the family  $\{A(t)\}_{t \in [0, T]}$  satisfies conditions  $(Hyp_1) - (Hyp_3)$  with  $V := D$  equipped with the norm given by  $\|v\|_V := \|A(0)v\| + \|v\|$  for  $v \in V$ .*

## 4 Averaging method for periodic solutions

We shall deal with the periodic problem

$$(P) \quad \begin{cases} \dot{u}(t) = A(t)u(t) + F(t, u(t)), & t \in [0, T] \\ u(0) = u(T) \end{cases}$$

where the family  $\{A(t)\}_{t \in [0, T]}$  of linear operators on a separable Banach space  $E$  satisfies a more restrictive variant of  $(Hyp_1)$  (from Proposition 3.6)

$(Hyp'_1)$  there is  $\omega > 0$ , such that

$$\|S_{A(t_1)}(s_1) \dots S_{A(t_n)}(s_n)\|_{\mathcal{L}(E, E)} \leq e^{-\omega(s_1 + \dots + s_n)};$$

whenever  $0 \leq t_1 \leq \dots \leq t_n \leq T$  and  $s_1, \dots, s_n \geq 0$ ,

conditions  $(Hyp_2)$ ,  $(Hyp_3)$  and, additionally,

$(Hyp_4)$  there is  $\mu_0 > -\omega$  such that the space  $(\mu_0 I - A_0)V$  is dense in  $E$ , where

$$A_0 := \frac{1}{T} \int_0^T A(\tau) d\tau \in \mathcal{L}(V, E);$$

$(Hyp_5)$   $A(0)x = A(T)x$  for  $x \in D(A(0)) = D(A(T))$ .

Furthermore, we assume that a continuous mapping  $F : [0, T] \times E \rightarrow E$

$(F_1)$  is locally Lipschitz with respect to the second variable uniformly with respect to the first one;

( $F_2$ ) has sublinear growth uniformly with respect to the first variable, i.e. there is a constant  $c > 0$  such that

$$\|F(t, x)\| \leq c(1 + \|x\|) \quad \text{for } x \in E, t \in [0, T];$$

( $F_3$ ) there is  $k \in [0, \omega)$  such that

$$\beta(F([0, T] \times Q)) \leq k\beta(Q) \quad \text{for any bounded } Q \subset E;$$

( $F_4$ )  $F(0, x) = F(T, x)$  for  $x \in E$ .

The existence of periodic solutions will be obtained by means of a continuation principle for a parameterized family of periodic problems

$$(P_{T, \lambda}) \quad \begin{cases} \dot{u}(t) = \lambda A(t)u(t) + \lambda F(t, u(t)), & t \in [0, T] \\ u(0) = u(T) \end{cases}$$

with the parameter  $\lambda \geq 0$ . For any  $x \in E$  and  $\lambda \in [0, T]$ , by  $u(\cdot; x, \lambda)$  denote the unique mild solution of

$$(22) \quad \dot{u}(t) = \lambda A(t)u(t) + \lambda F(t, u(t)), \quad t \in [0, T]$$

satisfying the initial condition  $u(0; x, \lambda) = x$ . The translation along trajectories operator for (22) is denoted by  $\Phi_t^{(\lambda)} : E \rightarrow E$  where  $t \in [0, T]$ . A point  $(x, \lambda) \in E \times [0, +\infty)$  is a  $T$ -periodic point for (22) if  $\Phi_T^{(\lambda)}(x) = x$ . We say that  $x_0 \in E$  is a *branching point* (or a *cobifurcation point*) for  $(P_{T, \lambda})$ ,  $\lambda \geq 0$ , if there exists a sequence of  $T$ -periodic points  $(x_n, \lambda_n) \in E \times (0, +\infty)$  such that  $\lambda_n \rightarrow 0$  and  $x_n \rightarrow x_0$  as  $n \rightarrow +\infty$ .

**Theorem 4.1** *If  $x_0 \in E$  is a branching point of  $(P_{T, \lambda})$ , then  $\hat{A}x_0 + \hat{F}(x_0) = 0$  where  $\hat{A} : D(\hat{A}) \rightarrow E$  is the closure of  $A_0$  and  $\hat{F} : E \rightarrow E$  is given by  $\hat{F}(x) := (1/T) \int_0^T F(\tau, x) d\tau$ .*

**Remark 4.2** (a) Recall that due to [14, Ch. 1, Th. 4.3], if  $A : D(A) \rightarrow E$  generates a  $C_0$  semigroup of contractions, then the dissipativity condition

$$(23) \quad \|x - \lambda Ax\| \geq \|x\| \quad \text{for any } x \in D(A), \lambda > 0$$

is equivalent to

$$(p, Ax) \leq 0 \quad \text{for any } x \in D(A), p \in J(x),$$

where  $J(x) := \{p \in E^* \mid \langle p, x \rangle = \|x\|^2 = \|p\|^2\}$  is the dual set of  $x$ .

(b) Hence, if  $(Hyp'_1)$  and  $(Hyp_3)$  hold, then  $\omega I + A_0 = \omega I + (1/T) \int_0^T A(\tau) d\tau$  in  $\mathcal{L}(V, E)$  is a dissipative operator. This implies that the closure  $\hat{A} : D(\hat{A}) \rightarrow E$  of  $A_0$  is a well-defined linear operator and, by (23), the operator  $\hat{A}_\omega := \omega I + \hat{A}$  is also dissipative, hence  $\lambda I - \hat{A}_\omega$  has closed range whenever  $\lambda > 0$ . If condition  $(Hyp_4)$  holds, then the operator  $(\mu_0 + \omega)I - \hat{A}_\omega = \mu_0 I - \hat{A}$  has closed and dense range, since  $\mu_0 + \omega > 0$ . It means that  $(\mu_0 + \omega)I - \hat{A}_\omega$  is  $m$ -dissipative, since its range is the whole  $E$  and, by the Lumer-Phillips theorem,  $\hat{A}$  generates a  $C_0$  semigroup such that  $\|S_{\hat{A}}(t)\| \leq e^{-\omega t}$  for  $t \geq 0$ .

(c) In particular,  $(-\omega, +\infty) \subset \varrho(\hat{A})$  and, for each  $\mu > -\omega$ ,

$$(\mu I - A_0)V = (\mu I - \hat{A})V = (\mu I - \hat{A})(\mu_0 I - \hat{A})^{-1}V_0$$

with  $V_0 := (\mu_0 I - \widehat{A})V$  being a dense subset of  $E$ . Since  $(\mu I - \widehat{A})(\mu_0 I - \widehat{A})^{-1} : E \rightarrow E$  is a bounded bijection, we infer that  $(\mu I - A_0)V$  is dense in  $E$  for any  $\mu > -\omega$ .

(d) If  $\{A(t)\}_{t \in [0, T]}$  satisfies conditions  $(Hyp'_1)$ ,  $(Hyp_2)$  and  $(Hyp_3)$ , then, in view of Proposition 3.6 and point (b), one has

$$\|R(t, s)\|_{\mathcal{L}(E, E)} \leq e^{-\omega(t-s)} \quad \text{for any } s, t \in [0, T], s \leq t.$$

In the proof of Theorem 4.1 and later on in the section we use the following lemma.

**Lemma 4.3** *Let  $\{A(t)\}_{t \in [0, T]}$  satisfy  $(Hyp'_1)$ ,  $(Hyp_2) - (Hyp_4)$  and let, for each  $\mu \in [0, 1]$ , the family  $\{A^{(\mu)}(t)\}_{t \in [0, T]}$  be defined by  $A^{(\mu)}(t) := -\mu I + (1 - \mu)A(t)$  for  $t \in [0, T]$ . Then, for each  $\lambda \geq 0$  and  $\mu \in [0, 1]$ , the family of operators  $\{\lambda A^{(\mu)}(t)\}_{t \in [0, T]}$  satisfies  $(Hyp_1)'$ ,  $(Hyp_2) - (Hyp_4)$  and the corresponding evolution systems  $\{R^{(\mu, \lambda)}(t, s)\}_{0 \leq s \leq t \leq T}$ ,  $\lambda \geq 0$ ,  $\mu \in [0, 1]$  have the following properties*

- (i) *for any  $x \in E$ ,  $t, s \in [0, T]$  with  $s \leq t$ ,  $(\lambda_n)$  in  $(0, +\infty)$  and  $(\mu_n)$  in  $[0, 1]$  such that  $\lambda_n \rightarrow 0$ ,  $\mu_n \rightarrow \mu_0$ , one has*

$$R^{(\mu_n, \lambda_n)}(t, s)x \rightarrow x \quad \text{as } n \rightarrow +\infty,$$

*uniformly with respect to  $t, s \in [0, T]$  with  $s \leq t$ ;*

- (ii) *if  $(k_n)$  is a sequence of positive integers and sequences  $(\lambda_n)$  in  $(0, +\infty)$  and  $(\mu_n)$  in  $[0, 1]$  are such that  $k_n \rightarrow +\infty$ ,  $k_n \lambda_n \rightarrow \varepsilon$  for some  $\varepsilon > 0$  and  $\mu_n \rightarrow \mu_0$  for some  $\mu_0 \in [0, 1]$ , then for any  $x \in E$*

$$R^{(\mu_n, \lambda_n)}(T, 0)^{k_n} x \rightarrow S_{\widehat{A^{(\mu_0)}}}(\varepsilon T)x \quad \text{as } n \rightarrow \infty,$$

*where  $\widehat{A^{(\mu_0)}}$  is the closure of the operator  $\frac{1}{T} \int_0^T A^{(\mu_0)}(\tau) d\tau$ ;*

- (iii) *if  $(k_n)$ ,  $(\lambda_n)$  and  $(\mu_n)$  are as in (ii), then for any  $x \in E$*

$$\lambda_n(I + R^{(\mu_n, \lambda_n)}(T, 0) + \dots + R^{(\mu_n, \lambda_n)}(T, 0)^{k_n-1})x \rightarrow \frac{1}{T} \int_0^{\varepsilon T} S_{\widehat{A^{(\mu_0)}}}(\tau)x d\tau \quad \text{as } n \rightarrow +\infty.$$

**Proof.** (i) It is easy to check that, for each  $\lambda > 0$  and  $\mu \in [0, 1]$ , the family  $\{\lambda A^{(\mu)}(t)\}_{t \in [0, T]}$  satisfies  $(Hyp'_1)$  with constant  $\omega := \lambda \min\{1, \omega\}$  and conditions  $(Hyp_2) - (Hyp_3)$  as well. From now on we write  $A_0^{(\mu)} := -\mu I + (1 - \mu)A_0$  for  $\mu \in [0, 1]$ . We claim that also  $(Hyp_4)$  holds. Indeed if  $\mu = 1$ , then  $A^{(\mu)}(t) = I$  for  $t \in [0, T]$  and  $(a_{\mu, \lambda}I - A_0^{(\mu)})V$  with  $a_{\mu, \lambda} = 0$  is dense in  $E$ , for  $\lambda > 0$ . If  $\mu \neq 1$ , then putting  $a_{\mu, \lambda} := (1 - \mu)\lambda\mu_0 - \lambda\mu$  we see that  $a_{\mu, \lambda} > -\omega$ , since  $\mu_0 > -\omega$ , and  $(a_{\mu, \lambda}I - \lambda A_0^{(\mu)})V = \lambda(1 - \mu)(\mu_0 I - A_0)V$  is dense in  $E$  and thus  $\lambda A_0^{(\mu)}$  satisfies  $(Hyp_4)$ . For the corresponding evolution system  $R^{(\mu, \lambda)}$ , one gets, for any  $(\lambda, \mu) \in (0, +\infty) \times [0, 1]$ ,

$$(24) \quad \|R^{(\mu, \lambda)}(t, s)\| \leq e^{-\lambda \bar{\omega}(t-s)} \leq 1 \quad \text{for } t, s \in [0, T], s \leq t,$$

with  $\bar{\omega} := \min\{1, \omega\}$ , and  $\frac{\partial}{\partial s} R^{(\mu, \lambda)}(t, s)v = -\lambda R^{(\mu, \lambda)}(t, s)A^{(\mu)}(s)v$ , for  $v \in V$ ,  $0 \leq s \leq t \leq T$ . In consequence, for any  $v \in V$ ,  $t, r \in [0, T]$ ,  $r \leq t$ ,  $\mu \in [0, 1]$  and  $\lambda > 0$ , one has

$$R^{(\mu, \lambda)}(t, r)v - v = R^{(\mu, \lambda)}(t, r)v - R^{(\mu, \lambda)}(t, t)v = \lambda \int_r^t R^{(\mu, \lambda)}(t, s)A^{(\mu)}(s)v ds.$$

Since, for any  $s \in [r, t]$ ,

$$\|R^{(\mu, \lambda)}(t, s)A^{(\mu)}(s)v\| \leq \|R^{(\mu, \lambda)}(t, s)\| \|A^{(\mu)}(s)\|_{\mathcal{L}(V, E)} \|v\|_V \leq \|A^{(\mu)}(s)\|_{\mathcal{L}(V, E)} \|v\|_V,$$

we infer that  $\|R^{(\mu, \lambda)}(t, r)v - v\| \leq \lambda C \|v\|_V$  with  $C := \sup_{\mu \in [0, 1]} \int_0^T \|A^{(\mu)}(s)\|_{\mathcal{L}(V, E)} ds < +\infty$ . This, due to the density of  $V$  in  $E$ , means that

$$\lim_{\lambda \rightarrow 0^+, \mu \rightarrow \mu_0} R^{(\mu, \lambda)}(t, r)x = x \quad \text{for any } x \in E$$

uniformly with respect to  $t, r \in [0, T]$  with  $r \leq t$ , which implies (i).

Define a map  $L : (0, +\infty) \times [0, 1] \rightarrow \mathcal{L}(E, E)$  by

$$L(\lambda, \mu) := R^{(\mu, \lambda)}(T, 0) \quad \text{for any } \lambda > 0 \text{ and } \mu \in [0, 1].$$

Clearly, by (24),  $\|L(\lambda, \mu)\| \leq 1$  for each  $(\lambda, \mu) \in (0, +\infty) \times [0, 1]$ . Observe also that, for each  $\mu \in [0, 1]$ ,  $a_\mu := a_{\mu, \lambda}/\lambda$  is such that  $(a_\mu I - A_0^{(\mu)})V$  is dense in  $E$ . Further, for any  $v \in V$ ,

$$(25) \quad \lambda^{-1}(L(\lambda, \mu)v - v) = \lambda^{-1}(R^{(\mu, \lambda)}(T, 0)v - v) = \int_0^T R^{(\mu, \lambda)}(T, s)A^{(\mu)}(s)v ds$$

and

$$\begin{aligned} \|\lambda^{-1}(L(\lambda, \mu)v - v) - TA_0^{(\mu_0)}v\| &\leq \int_0^T \|R^{(\mu, \lambda)}(T, s)A^{(\mu)}(s)v - A^{(\mu_0)}(s)v\| ds \\ &\leq \int_0^T \|R^{(\mu, \lambda)}(T, s)A^{(\mu)}(s)v - A^{(\mu)}(s)v\| ds + \|v\|_V \int_0^T \|A^{(\mu)}(s) - A^{(\mu_0)}(s)\|_{\mathcal{L}(V, E)} ds, \end{aligned}$$

which by use of point (i) of this lemma and (*Hyp*<sub>3</sub>) gives

$$\lim_{\lambda \rightarrow 0^+, \mu \rightarrow \mu_0} \lambda^{-1}(L(\lambda, \mu)v - v) = TA_0^{(\mu_0)}v \quad \text{for } v \in V.$$

Hence, applying Theorem 2.1 and changing the time variable we get (ii) and (iii) as the closure of  $A_0^{(\mu_0)}$  is equal to  $\widehat{A^{(\mu_0)}}$ .  $\square$

**Proof of Theorem 4.1.** Let sequences  $(\lambda_n)$  in  $(0, +\infty)$  and  $(x_n)$  in  $E$  be such that  $\lambda_n \rightarrow 0$ ,  $x_n \rightarrow x_0$  as  $n \rightarrow \infty$  and  $\Phi_T^{(\lambda_n)}(x_n) = x_n$  for each  $n \geq 1$ . Then, by definition

$$(26) \quad \Phi_t^{(\lambda_n)}(x_n) = R^{(\lambda_n)}(t, 0)x_n + \lambda_n \int_0^t R^{(\lambda_n)}(t, s)F(s, \Phi_s^{(\lambda_n)}(x_n)) ds$$

for any  $t \in [0, T]$ , where  $\{R^{(\lambda)}(t, s)\}_{0 \leq s \leq t \leq T}$  denotes the evolution system generated by the family  $\{\lambda A(t)\}_{t \in [0, T]}$ . This yields

$$\begin{aligned} \|x_n - \Phi_t^{(\lambda_n)}(x_n)\| &= \|R^{(\lambda_n)}(t, 0)x_n - x_n\| + \left\| \lambda_n \int_0^t R^{(\lambda_n)}(t, s)F(s, \Phi_s^{(\lambda_n)}(x_n)) ds \right\| \\ &\leq \|R^{(\lambda_n)}(t, 0)x_n - x_n\| + \lambda_n c \int_0^t (1 + \|\Phi_s^{(\lambda_n)}(x_n)\|) ds. \end{aligned}$$

In view of Lemma 4.3 (i) with  $\mu_n := 0$  for  $n \geq 1$  and the boundedness of  $\{\Phi_s^{(\lambda_n)}(x_n) \mid s \in [0, T], n \geq 1\}$ , we infer that  $\Phi_t^{(\lambda_n)}(x_n) \rightarrow x_0$ , uniformly with respect to  $t \in [0, T]$ . Further, by (26), one has

$$(27) \quad x_n = \Phi_T^{(\lambda_n)}(x_n) = R^{(\lambda_n)}(T, 0)x_n + \lambda_n \int_0^T R^{(\lambda_n)}(T, s)F(s, \Phi_s^{(\lambda_n)}(x_n)) ds$$

and consequently, for each  $k \geq 0$

$$R^{(\lambda_n)}(T, 0)^k x_n = R^{(\lambda_n)}(T, 0)^{k+1} x_n + \lambda_n R^{(\lambda_n)}(T, 0)^k \int_0^T R^{(\lambda_n)}(T, s)F(s, \Phi_s^{(\lambda_n)}(x_n)) ds.$$

Let  $\varepsilon > 0$  be arbitrary and let  $(k_n)$  be a sequence of positive integers such that  $k_n \lambda_n \rightarrow \varepsilon$ . Summing up the above equalities with  $k = 0, 1, \dots, k_n - 1$  for any  $n \geq 1$ , we obtain

$$x_n = R^{(\lambda_n)}(T, 0)^{k_n} x_n + \lambda_n \left[ \sum_{k=0}^{k_n-1} R^{(\lambda_n)}(T, 0)^k \right] \left( \int_0^T R^{(\lambda_n)}(T, s)F(s, \Phi_s^{(\lambda_n)}(x_n)) ds \right)$$

and, by use of Lemma 4.3 (ii) and (iii) with  $\mu_n := 0$  for  $n \geq 1$ ,

$$x_0 = S_{\hat{A}}(\varepsilon T)x_0 + \left[ \frac{1}{T} \int_0^{\varepsilon T} S_{\hat{A}}(\tau) d\tau \right] \left( \int_0^T F(s, x_0) ds \right).$$

In consequence

$$-\frac{1}{\varepsilon T} (S_{\hat{A}}(\varepsilon T)x_0 - x_0) = \frac{1}{\varepsilon T} \int_0^{\varepsilon T} S_{\hat{A}}(\tau) \hat{F}(x_0) d\tau.$$

Thus, since  $\varepsilon > 0$  was arbitrary, letting  $\varepsilon \rightarrow 0^+$ , one has  $-\hat{A}x_0 = \hat{F}(x_0)$ .  $\square$

**Theorem 4.4** (Averaging principle) *Let  $\{A(t)\}_{t \in [0, T]}$  be a family of generators of  $C_0$  semigroups satisfying  $(Hyp'_1)$ ,  $(Hyp_2)$  –  $(Hyp_5)$  and let  $F : [0, T] \times E \rightarrow E$  be a continuous map with properties  $(F_1)$  –  $(F_4)$ . If  $U \subset E$  is an open bounded set such that  $\hat{A}x + \hat{F}(x) \neq 0$  for any  $x \in \partial U \cap D(\hat{A})$ , then there exists  $\lambda_0 > 0$  such that for all  $\lambda \in (0, \lambda_0]$ ,  $\Phi_T^{(\lambda)}(x) \neq x$  for all  $x \in \partial U$  and*

$$(28) \quad \text{Deg}(\hat{A} + \hat{F}, U) = \text{deg}(I - \Phi_T^{(\lambda)}, U).$$

Here  $\text{deg}$  stands for the topological degree for condensing vector fields (see [1] or [11]) and  $\text{Deg}(\hat{A} + \hat{F}, U) := \text{deg}(I + \hat{A}^{-1}\hat{F}, U)$  (see [4]).

In the proof we shall need the following lemmata.

**Lemma 4.5** (see [4, Lemma 5.4]) *Let  $T_n : E \rightarrow E$ ,  $n \geq 1$ , be bounded linear operators, such that, for any  $x \in E$ ,  $(T_n x)$  is a Cauchy sequence <sup>(3)</sup>. Then, for any bounded set  $\{x_n\}_{n \geq 1} \subset E$*

$$\beta(\{T_n x_n\}_{n \geq 1}) \leq \left( \limsup_{n \rightarrow +\infty} \|T_n\| \right) \beta(\{x_n\}_{n \geq 1}).$$

---

<sup>3</sup>This is actually equivalent to the existence of a bounded operator  $T : E \rightarrow E$  being the strong limit of  $(T_n)$ .



**Lemma 4.6** (cf. Step 2 in the proof of Theorem 5.1 in [4]) *Let  $A$  be a generator of a  $C_0$  semigroup  $S_A$  such that  $\|S_A(t)\| \leq e^{-\omega t}$  for  $t \geq 0$  and  $F : E \rightarrow E$  be a continuous map with  $k \in [0, \omega)$  such that  $\beta(F(Q)) \leq k\beta(Q)$  for any bounded  $Q$ . If an open bounded  $U \subset E$  is such that  $Ax + F(x) \neq 0$  for each  $x \in \partial U \cap D(A)$ , then there exists a locally Lipschitz compact mapping  $F_L : E \rightarrow E$  such that*

$$(29) \quad Ax + (1 - \mu)F(x) + \mu F_L(x) \neq 0 \quad \text{for } x \in \partial U \cap D(A), \mu \in [0, 1].$$

**Lemma 4.7** *Let  $\{A^{(\mu)}(t)\}_{t \in [0, T]}$  for  $\mu \in [0, 1]$ , satisfy  $(Hyp'_1)$ ,  $(Hyp_2) - (Hyp_5)$  with the common, independent of  $\mu$ , constants  $\omega > 0$ ,  $\omega_V$ ,  $M_V$  and let  $F : [0, T] \times E \times [0, 1] \rightarrow E$  be a continuous mapping satisfying  $(F_1)_{par} - (F_3)_{par}$  and the periodicity condition*

$$F(0, x, \lambda) = F(T, x, \lambda) \quad \text{for } (x, \lambda) \in E \times [0, 1].$$

*Suppose that  $U \subset E$  is open bounded and  $\widehat{A^{(\mu)}}x + \widehat{F}(x, \mu) \neq 0$  for  $x \in \partial U \cap D(\widehat{A^{(\mu)}})$  and  $\mu \in [0, 1]$ , where  $\widehat{A^{(\mu)}}$  is the closure of  $(1/T) \int_0^T A^{(\mu)}(s) ds$  and  $\widehat{F} : E \times [0, 1] \rightarrow E$  is given by  $\widehat{F}(x, \mu) := (1/T) \int_0^T F(s, x, \mu) ds$ . Then, there exists  $\lambda_0 > 0$ , such that, for any  $\lambda \in (0, \lambda_0]$ ,*

$$\Psi_T^{(\lambda)}(x, \mu) \neq x \quad \text{for all } x \in \partial U, \mu \in [0, 1],$$

*where  $\Psi_T^{(\lambda)} : \overline{U} \times [0, 1] \rightarrow E$  is given by  $\Psi_T^{(\lambda)}(x, \mu) := u(T; x, \mu, \lambda)$  for  $(x, \mu) \in \overline{U} \times [0, 1]$ ,  $\lambda > 0$  and  $u(\cdot; x, \mu, \lambda) : [0, T] \rightarrow E$  is the unique mild solution of*

$$\begin{cases} \dot{u}(t) = \lambda A^{(\mu)}(t)u(t) + \lambda F(t, u(t), \mu), & t \in [0, T] \\ u(0) = x. \end{cases}$$

**Proof.** Suppose to the contrary that there exist sequences  $(\lambda_n)$  in  $(0, +\infty)$  with  $\lambda_n \rightarrow 0^+$ ,  $(x_n)$  in  $\partial U$  and  $(\mu_n)$  in  $[0, 1]$  such that  $\Psi_T^{(\lambda_n)}(x_n, \mu_n) = x_n$  for  $n \geq 1$ . Without loss of generality, we may assume that  $\mu_n \rightarrow \mu_0$  as  $n \rightarrow +\infty$ , for some  $\mu_0 \in [0, 1]$ . Let  $\{\overline{A^{(\mu)}}(t)\}_{t \in [0, 2T]}$ ,  $\mu \in [0, 1]$  and a mapping  $\overline{F} : [0, 2T] \times E \times [0, 1] \rightarrow E$  be given by

$$\begin{aligned} \overline{A^{(\mu)}}(t) &:= A^{(\mu)}(t - [t/T]T) & \text{for } (t, \mu) \in [0, 2T] \times [0, 1], \\ \overline{F}(t, x, \mu) &:= F(t - [t/T]T, x, \mu) & \text{for } (t, x, \mu) \in [0, 2T] \times E \times [0, 1]. \end{aligned}$$

where  $[s]$  stands for the integer part of  $s \in \mathbb{R}$ . It is easy to check that, for each  $\lambda \in (0, \infty)$  and  $\mu \in [0, 1]$ , the family  $\{\lambda \overline{A^{(\mu)}}(t)\}_{t \in [0, 2T]}$  and the mapping  $\lambda \overline{F}$  satisfies  $(Hyp'_1)$ ,  $(Hyp_2) - (Hyp_4)$  and  $(F1)_{par} - (F3)_{par}$ . Denote by  $\{\overline{R^{(\mu, \lambda)}}\}_{0 \leq s \leq t \leq 2T}$  the corresponding evolution system obtained by Proposition 3.6. From the very construction of hyperbolic evolution systems (see [14, Ch. 5, Theorem 3.1] and the proof of Proposition 3.7), we see that, for all  $\lambda > 0$ , and  $\mu \in [0, 1]$ ,

$$(30) \quad \overline{R^{(\mu, \lambda)}}(T + t, T + s) = \overline{R^{(\mu, \lambda)}}(t, s) = R^{(\mu, \lambda)}(t, s) \quad \text{for } t, s \in [0, T], s \leq t.$$

For each  $t \in [0, 2T]$ , define  $\overline{\Psi}_t^{(\lambda)} : E \times [0, 1] \rightarrow E$ , by  $\overline{\Psi}_t^{(\lambda)}(x, \mu) := \overline{u}(t; x, \mu, \lambda)$ , where  $\overline{u}(\cdot; x, \mu, \lambda) : [0, 2T] \rightarrow E$  is a solution of

$$\begin{cases} \dot{u}(t) = \lambda \overline{A^{(\mu)}}(t)u(t) + \lambda \overline{F}(t, u(t), \mu), & t \in [0, 2T] \\ u(0) = x. \end{cases}$$

It clearly follows from (30) that, for any  $n \geq 1$  and  $t \in [0, T]$ ,

$$\Psi_t^{(\lambda_n)}(x_n, \mu_n) = \overline{\Psi}_{t+T}^{(\lambda_n)}(x_n, \mu_n) = \overline{R}^{(\mu_n, \lambda_n)}(T+t, t) \Psi_t^{(\lambda_n)}(x_n, \mu_n) + \lambda_n \int_t^{T+t} w_{n,t}(s) ds,$$

with  $w_{n,t}(s) := \overline{R}^{(\mu_n, \lambda_n)}(T+t, s) \overline{F}(s, \overline{\Psi}_s^{(\lambda_n)}(x_n, \mu_n), \mu_n)$  for  $s \in [t, T+t]$ . Therefore, for any integer  $k \geq 0$ , one has

$$\begin{aligned} \overline{R}^{(\mu_n, \lambda_n)}(T+t, t)^k \Psi_t^{(\lambda_n)}(x_n, \mu_n) &= \overline{R}^{(\mu_n, \lambda_n)}(T+t, t)^{k+1} \Psi_t^{(\lambda_n)}(x_n, \mu_n) \\ &\quad + \lambda_n \overline{R}^{(\mu_n, \lambda_n)}(T+t, t)^k \int_t^{T+t} w_{n,t}(s) ds. \end{aligned}$$

Putting  $k_n := [1/\lambda_n]$  and summing up the above equalities with  $k = 0, \dots, k_n - 1$ , we find that

$$(31) \quad \Psi_t^{(\lambda_n)}(x_n, \mu_n) = \overline{R}^{(\mu_n, \lambda_n)}(T+t, t)^{k_n} \Psi_t^{(\lambda_n)}(x_n, \mu_n) + K_n \left( \int_t^{T+t} w_{n,t}(s) ds \right)$$

where

$$K_n := \lambda_n \sum_{k=0}^{k_n-1} \overline{R}^{(\mu_n, \lambda_n)}(T+t, t)^k \quad \text{for } n \geq 1.$$

By (30) and the fact that  $\lambda_n k_n \rightarrow 1$  as  $n \rightarrow +\infty$ , going along the lines of the proof of Lemma 4.3, we infer that, for any  $x \in E$ ,

$$\overline{R}^{(\mu_n, \lambda_n)}(T+t, t)^{k_n} x \rightarrow S_{\widehat{A(\mu_0)}}(T)x \quad \text{and} \quad K_n x \rightarrow \frac{1}{T} \int_0^T S_{\widehat{A(\mu_0)}}(s)x ds.$$

This, along with Lemmata 4.5 and 3.4, gives

$$\begin{aligned} \beta \left( \left\{ \Psi_t^{(\lambda_n)}(x_n, \mu_n) \right\}_{n \geq 1} \right) &\leq \\ &\leq e^{-\omega T} \beta \left( \left\{ \Psi_t^{(\lambda_n)}(x_n, \mu_n) \right\}_{n \geq 1} \right) + \frac{1 - e^{-\omega T}}{\omega T} \beta \left( \left\{ \int_t^{T+t} w_{n,t}(s) ds \right\}_{n \geq 1} \right) \\ &\leq e^{-\omega T} \beta \left( \left\{ \Psi_t^{(\lambda_n)}(x_n, \mu_n) \right\}_{n \geq 1} \right) + \frac{1 - e^{-\omega T}}{\omega T} \int_t^{T+t} \beta(\{w_{n,t}(s)\}_{n \geq 1}) ds \\ &\leq e^{-\omega T} \beta \left( \left\{ \Psi_t^{(\lambda_n)}(x_n, \mu_n) \right\}_{n \geq 1} \right) + \frac{1 - e^{-\omega T}}{\omega T} \int_t^{T+t} k\beta \left( \left\{ \overline{\Psi}_s^{(\lambda_n)}(x_n, \mu_n) \right\}_{n \geq 1} \right) ds \end{aligned}$$

and, in consequence,

$$(32) \quad \beta \left( \left\{ \Psi_t^{(\lambda_n)}(x_n, \mu_n) \right\}_{n \geq 1} \right) \leq \frac{k}{\omega T} \int_t^{T+t} \beta \left( \left\{ \overline{\Psi}_s^{(\lambda_n)}(x_n, \mu_n) \right\}_{n \geq 1} \right) ds.$$

Define  $\phi : [0, 2T] \rightarrow \mathbb{R}$  by

$$\phi(s) := \beta \left( \left\{ \overline{\Psi}_s^{(\lambda_n)}(x_n, \mu_n) \right\}_{n \geq 1} \right) \quad \text{for } s \in [0, 2T].$$

We claim that  $\phi \equiv 0$ . Indeed, otherwise  $M := \sup_{s \in [0, 2T]} \phi(s) \in (0, +\infty)$  and by its  $T$ -periodicity, for  $\varepsilon \in (0, (1 - k/\omega)M)$  there exists  $t_\varepsilon \in [0, T]$  such that

$$M - \varepsilon < \phi(t_\varepsilon) \leq \frac{k}{\omega T} \int_{t_\varepsilon}^{T+t_\varepsilon} \phi(s) ds \leq (k/\omega)M < M - \varepsilon,$$

which is a contradiction. Hence, in particular  $\beta(\{x_n\}_{n \geq 1}) = 0$  and without loss of generality we may assume that  $x_n \rightarrow x_0$  as  $n \rightarrow +\infty$ , for some  $x_0 \in \partial U$ .

Further, observe that  $\Psi_t^{(\lambda_n)}(x_n, \mu_n) \rightarrow x_0$  as  $n \rightarrow +\infty$ , uniformly with respect to  $t \in [0, T]$ , which follows from the inequality

$$\|\Psi_t^{(\lambda_n)}(x_n, \mu_n) - x_0\| \leq \|R^{(\mu_n, \lambda_n)}(t, 0)x_n - x_0\| + \lambda_n \int_0^t \|R^{(\mu_n, \lambda_n)}(t, s)F(s, \Psi_s^{(\lambda_n)}(x_n, \mu_n), \mu_n)\| ds$$

and Lemma 4.3 (i). Note that, for any  $k \geq 0$

$$R^{(\mu_n, \lambda_n)}(T, 0)^k x_n = R^{(\mu_n, \lambda_n)}(T, 0)^{k+1} x_n + \lambda_n R^{(\mu_n, \lambda_n)}(T, 0)^k \left( \int_0^T h_n(s) ds \right),$$

where  $h_n(s) := R^{(\mu_n, \lambda_n)}(T, s)F(s, \Psi_s^{(\lambda_n)}(x_n, \mu_n), \mu_n)$  for  $s \in [0, T]$  and  $n \geq 1$ . Let  $\varepsilon > 0$  be arbitrary and a sequence  $(k_n)$  of positive integers be such that  $k_n \rightarrow +\infty$  and  $k_n \lambda_n \rightarrow \varepsilon$  as  $n \rightarrow +\infty$ . Then, reasoning as before, one obtains

$$(33) \quad x_n = R^{(\mu_n, \lambda_n)}(T, 0)^{k_n} x_n + J_n \left( \int_0^T h_n(s) ds \right) \quad \text{for } n \geq 1,$$

where  $J_n := \lambda_n \sum_{k=0}^{k_n-1} R^{(\mu_n, \lambda_n)}(T, 0)^k$ . Note that, in view of Lemma 4.3,

$$\begin{aligned} h_n(s) &\rightarrow F(s, x_0, \mu_0) \quad \text{as } n \rightarrow +\infty, \text{ uniformly for } s \in [0, T], \\ R^{(\mu_n, \lambda_n)}(T, 0)^{k_n} x_n &\rightarrow S_{\widehat{A(\mu_0)}}(\varepsilon T)x_0 \quad \text{as } n \rightarrow +\infty \text{ and} \\ J_n x &\rightarrow \frac{1}{T} \int_0^{\varepsilon T} S_{\widehat{A(\mu_0)}}(s)x ds \quad \text{as } n \rightarrow +\infty, \text{ for any } x \in E. \end{aligned}$$

Thus, after passing in (33) to the limit with  $n \rightarrow +\infty$ , one has

$$x_0 = S_{\widehat{A(\mu_0)}}(\varepsilon T)x_0 + \left[ \frac{1}{T} \int_0^{\varepsilon T} S_{\widehat{A(\mu_0)}}(\tau) d\tau \right] \left( \int_0^T F(s, x_0, \mu_0) ds \right),$$

which rewritten as

$$-\frac{1}{\varepsilon T} \left( S_{\widehat{A(\mu_0)}}(\varepsilon T)x_0 - x_0 \right) = \frac{1}{\varepsilon T} \int_0^{\varepsilon T} S_{\widehat{A(\mu_0)}}(\tau) \widehat{F}(x_0, \mu_0) d\tau.$$

Letting  $\varepsilon \rightarrow 0^+$  yields  $-\widehat{A(\mu_0)}x_0 = \widehat{F}(x_0, \mu_0)$ , a contradiction completing the proof.  $\square$

**Lemma 4.8** (see [2, Proposition 4.3]) *Let  $F : E \rightarrow E$  be a completely continuous locally Lipschitz with sublinear growth and let  $\Xi_t : E \rightarrow E$  be the translation along trajectories operator by time  $t > 0$  for the equation*

$$\dot{u}(t) = -u(t) + F(u(t)), \quad t \in [0, T].$$

*Then, for each  $t > 0$ , the mapping  $\Xi_t$  is a  $k$ -set contraction and if an open bounded  $U \subset E$  is such that  $0 \notin (I - F)(\partial U)$ , then there exists  $t_0 > 0$  such that for any  $t \in (0, t_0]$ ,  $\Xi_t(x) \neq x$  and*

$$\deg(I - F, U) = \deg(I - \Xi_t, U).$$

**Proof of Theorem 4.4.** First we reduce the proof to the case where the nonlinear perturbation is compact. By Remark 4.2 (b) we infer that the operator  $\widehat{A}$  and mapping  $\widehat{F}$  satisfy assumptions of Lemma 4.6. Therefore there is locally Lipschitz compact mapping  $\widehat{F}_L : E \rightarrow E$  such that

$$(34) \quad \widehat{A}x + (1 - \mu)\widehat{F}(x) + \mu\widehat{F}_L(x) \neq 0 \quad \text{for } x \in \partial U \cap D(\widehat{A}), \mu \in [0, 1].$$

Thus, applying Lemma 4.7 to equations associated to

$$\dot{u}(t) = \lambda A(t)u(t) + \lambda((1 - \mu)\widehat{F}(u(t)) + \mu\widehat{F}_L(u(t))), \quad t \in [0, T],$$

and the homotopy invariance of the topological degree, provide

**Claim A.** *There exists  $\lambda_1 > 0$  such that for any  $\lambda \in (0, \lambda_1]$*

$$\deg(I - \Phi_T^{(\lambda)}, U) = \deg(I - \widetilde{\Phi}_T^{(\lambda)}, U),$$

where  $\widetilde{\Phi}_T^{(\lambda)} : \overline{U} \rightarrow E$  is the translation along trajectories operator by the time  $T$  for the equation

$$\dot{u}(t) = \lambda A(t)u(t) + \lambda\widehat{F}_L(u(t)).$$

Next we prove

**Claim B.** *There exists  $\lambda_2 \in (0, \lambda_1]$  such that for any  $\lambda \in (0, \lambda_2]$*

$$(35) \quad \deg(I - \widetilde{\Phi}_T^{(\lambda)}, U) = \deg(I - \overline{\Phi}_T^{(\lambda)}, U),$$

where  $\overline{\Phi}_T^{(\lambda)}$  is the translation along trajectories operator by the time  $T$  for the equation

$$\dot{u}(t) = -\lambda u(t) - \lambda\widehat{A}^{-1}\widehat{F}_L(u(t)), \quad t \in [0, T].$$

To this end, consider a differential problem given by

$$(36) \quad \dot{u}(t) = \lambda\widetilde{A}^{(\mu)}(t)u(t) + \lambda\widetilde{F}(u(t), \mu) \quad \text{on } [0, T],$$

where

$$\begin{aligned} \widetilde{A}^{(\mu)}(t) &:= -\mu I + (1 - \mu)A(t) & \text{for } t \in [0, T], \\ \widetilde{F}(x, \mu) &:= [(1 - \mu)I - \mu\widehat{A}^{-1}]\widehat{F}_L(x) & \text{for } x \in E, \mu \in [0, 1]. \end{aligned}$$

Lemma 4.3 shows that the family  $\{\lambda\widetilde{A}^{(\mu)}(t)\}_{t \in [0, T]}$  satisfies  $(Hyp'_1)$ ,  $(Hyp_2) - (Hyp_4)$  and so the family  $\{\widetilde{A}^{(\mu)}(t)\}_{t \in [0, T]}$  fulfills the assumptions of Lemma 4.7. It is also clear that  $\widetilde{F}$  is locally Lipschitz in  $x$  uniformly with respect to  $\mu$  and compact, which, in particular, means that it has sublinear growth uniformly with respect to  $\mu$ . For any  $\lambda \in (0, \infty)$ , let  $\Psi : \overline{U} \times [0, 1] \rightarrow E$  be given by

$$\Psi_T^{(\lambda)}(x, \mu) := u(T; x, \mu, \lambda), \quad \text{for } x \in E, \mu \in [0, 1],$$

where  $u(\cdot; x, \mu, \lambda)$  stands for the mild solution of (36) starting at  $x$ .

Observe that

$$(37) \quad [(1 - \mu)\widehat{A} - \mu I]x + \widetilde{F}(x, \mu) \neq 0 \quad \text{for } \mu \in [0, 1], x \in \partial U \cap D(\widehat{A}^{(\mu)}).$$

Indeed, suppose that for some  $\mu \in [0, 1]$  and  $x \in \partial U \cap D(\hat{A})$  we have  $[(1 - \tilde{\mu})\hat{A} - \mu I]x + \tilde{F}(x, \mu) = 0$ . If  $\mu = 1$  then  $-x - \hat{A}^{-1}\hat{F}_L(x) = 0$ , which contradicts (34) and if  $\mu \in [0, 1)$  then

$$x = -R(\hat{A}; \mu/(1 - \mu))(I - \mu/(1 - \mu)\hat{A}^{-1})\hat{F}_L(x),$$

which due to the resolvent identity gives  $x = -\hat{A}^{-1}\hat{F}_L(x)$ , again a contradiction proving (37). Thus, applying Lemma 4.7 and the homotopy invariance of the topological degree to  $\Psi_T^{(\lambda)}$ , we find  $\lambda_2 \in (0, \lambda_1]$  such that for any  $\lambda \in (0, \lambda_2]$ , (35) holds, which ends the proof of Claim B.

Finally, by applying Lemma 4.8, one gets  $\lambda_0 \in (0, \lambda_2]$  such that for any  $\lambda \in (0, \lambda_0]$

$$\deg(I + \hat{A}^{-1}\hat{F}_L, U) = \deg(I - \overline{\Phi}_{\lambda T}^{(1)}, U).$$

Combining this with (34), we infer that, for  $\lambda \in (0, \lambda_0]$ ,

$$\begin{aligned} \text{Deg}(\hat{A} + \hat{F}, U) &= \text{Deg}(\hat{A} + \hat{F}_L, U) = \deg(I + \hat{A}^{-1}\hat{F}_L, U) \\ &= \deg(I - \overline{\Phi}_{\lambda T}^{(1)}, U) = \deg(I - \overline{\Phi}_T^{(\lambda)}, U), \end{aligned}$$

which together with Claims A and B completes the proof.  $\square$

As an immediate consequence of Theorem 4.4, we get the following result.

**Corollary 4.9** *If  $0 \notin (\hat{A} + \hat{F})(\partial U \cap D(\hat{A}))$  and  $\deg(\hat{A} + \hat{F}, U) \neq 0$ , then  $(P_{T, \lambda})$  admits a solution for small  $\lambda > 0$ .*

By means of *a priori bounds* type assumption, we get the existence criterion for periodic solutions.

**Theorem 4.10** (Continuation principle) *Let a family  $\{A(t)\}_{t \in [0, T]}$  and a mapping  $F : [0, T] \times E \rightarrow E$  satisfy  $(\text{Hyp}'_1)$ ,  $(\text{Hyp}_2) - (\text{Hyp}_5)$  and  $(F_1) - (F_3)$ , respectively. If  $(P_{T, \lambda})$  has no  $T$ -periodic points in  $\partial U \times (0, 1)$  and  $\text{Deg}(\hat{A} + \hat{F}, U) \neq 0$ , then  $(P)$  admits a mild solution  $u : [0, T] \rightarrow E$  such that  $u(0) = u(T) \in \overline{U}$ .*

**Proof.** If  $\Phi_T^{(1)}(x) = x$  for some  $x \in \partial U$ , then the assertion holds. Hence, assume that  $\Phi_T^{(1)}(x) \neq x$  for  $x \in \partial U$ . By Theorem 4.4, there exists  $\lambda_0 \in (0, 1)$  such that, for any  $\lambda \in (0, \lambda_0]$ ,  $\Phi_T^{(\lambda)}(x) \neq x$  and

$$(38) \quad \deg(I - \Phi_T^{(\lambda)}, U) = \text{Deg}(\hat{A} + \hat{F}, U).$$

Then the mapping  $\overline{U} \times [\lambda_0, 1] \ni (x, \lambda) \mapsto \Phi_T^{(\lambda)}(x)$  provides an admissible homotopy (in the degree theory of  $k$ -set contraction vector fields) and by the homotopy invariance

$$\deg(I - \Phi_T^{(1)}, U) = \deg(I - \Phi_T^{(\lambda_0)}, U),$$

which together with (38) and the assumption implies the existence of  $x \in U$  such that  $\Phi_T^{(1)}(x) = x$ .  $\square$

The above Continuation Principle can be useful when studying asymptotically linear evolution systems.

**Theorem 4.11** *Let a family  $\{A(t)\}_{t \in [0, T]}$  satisfy  $(Hyp'_1)$  and  $(Hyp_2) - (Hyp_5)$  and let  $F : [0, T] \times E \rightarrow E$  be a completely continuous mapping satisfying  $(F_1)$ ,  $(F_2)$  and  $(F_4)$ . Assume also that  $\{F_\infty(t) : E \rightarrow E\}_{t \in [0, T]}$  is a family of compact linear operators such that the mapping  $t \mapsto F_\infty(t) \in \mathcal{L}(E, E)$  is continuous on  $[0, T]$ ,  $F_\infty(0) = F_\infty(T)$  and*

$$(39) \quad \lim_{\|x\| \rightarrow +\infty} \frac{\|F(t, x) - F_\infty(t)x\|}{\|x\|} = 0 \quad \text{uniformly with respect to } t \in [0, T].$$

*If, for each  $\lambda \in (0, 1]$ , the parameterized linear periodic problem*

$$(40) \quad \begin{cases} \dot{u}(t) = \lambda(A(t) + F_\infty(t))u(t), & t \in [0, T] \\ u(0) = u(T) \end{cases}$$

*has no nontrivial solution and  $\text{Ker}(\hat{A} + \hat{F}_\infty) = \{0\}$ , then  $(P)$  admits a  $T$ -periodic mild solution.*

**Proof.** We begin with proving that there exists  $R_1 > 0$  such that  $0 \notin (\hat{A} + \hat{F})(E \setminus B(0, R_1)) \cap D(\hat{A})$  and

$$(41) \quad \left| \text{Deg}(\hat{A} + \hat{F}, B(0, R_1)) \right| = 1.$$

Define  $H : [0, T] \times E \times [0, 1] \rightarrow E$  by

$$H(t, x, \lambda) := \begin{cases} \lambda F(t, \lambda^{-1}x) & \text{for } t \in [0, T], x \in E, \lambda \in (0, 1], \\ F_\infty(t)x & \text{for } t \in [0, T], x \in E, \lambda = 0. \end{cases}$$

Standard arguments show that both  $H$  and  $\hat{H}$  are completely continuous and for any  $\lambda_0 \in [0, 1]$

$$(42) \quad \lim_{\|x\| \rightarrow +\infty, \lambda \rightarrow \lambda_0} \frac{\|\hat{H}(x, \lambda) - \hat{F}_\infty x\|}{\|x\|} = 0.$$

Observe that there is  $R_1 > 0$  such that

$$(43) \quad \hat{A}x + \hat{H}(x, \lambda) \neq 0 \quad \text{for } x \in (E \setminus B(0, R_1)) \cap D(\hat{A}), \lambda \in [0, 1].$$

Otherwise there are  $(x_n)$  in  $E$  and  $(\lambda_n)$  in  $[0, 1]$  such that  $\hat{A}x_n + \hat{H}(x_n, \lambda_n) = 0$  and  $\|x_n\| \rightarrow +\infty$ . Put  $z_n := x_n / \|x_n\|$ . If  $\lambda_n = 0$  for some  $n \geq 1$ , then  $z_n = -\hat{A}^{-1}\hat{F}_\infty z_n$ , a contradiction to the assumption. If  $(\lambda_n)$  in  $(0, 1]$ , then

$$(44) \quad z_n = -\|x_n\|^{-1} \hat{A}^{-1} \hat{H}(x_n, \lambda_n) = -\hat{A}^{-1}(\lambda_n^{-1} \|x_n\|)^{-1} \hat{F}(\lambda_n^{-1} \|x_n\| z_n).$$

Since  $\lim_{\|x\| \rightarrow +\infty} \|\hat{F}(x) - \hat{F}_\infty x\| / \|x\| = 0$  and  $\rho_n := \lambda_n^{-1} \|x_n\| \rightarrow +\infty$  as  $n \rightarrow \infty$ ,

$$(45) \quad \|z_n + \hat{A}^{-1} \hat{F}_\infty z_n\| \leq \|\hat{A}^{-1}\| \|\hat{F}(\rho_n z_n) - \hat{F}_\infty(\rho_n z_n)\| / \rho_n \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By Lemma 3.4 the linear operator  $\hat{F}_\infty$  is compact, which together with (45) means that  $(z_n)$  contains a convergent subsequence. Hence, we may assume that  $z_n \rightarrow z_0$  for some  $z_0 \in E$  and by (45) we infer that  $z_0 = -\hat{A}^{-1} \hat{F}_\infty z_0$ , which is again a contradiction meaning

that (41) holds for sufficiently large  $R_1 > 0$ .

Now we claim that

(46) there is  $R \geq R_1$  such that, for any  $\lambda \in (0, 1)$ , the problem  $(P_{T,\lambda})$  has no periodic solutions starting at points from  $\partial B(0, R)$ .

Otherwise there exist  $(u_n)$  and  $(\lambda_n)$  in  $(0, 1)$  such that, for each  $n \geq 1$ ,  $u_n$  is a solution of  $(P_{T,\lambda_n})$  and  $\|u_n(0)\| \rightarrow +\infty$  as  $n \rightarrow +\infty$ . Putting  $v_n := u_n/\|u_n\|_\infty$ , where  $\|u_n\|_\infty := \max_{t \in [0, T]} \|u_n(t)\|$ , one has

$$(47) \quad v_n(t) = R^{(\lambda_n)}(t, 0)v_n(0) + \lambda_n \|u_n\|_\infty^{-1} \int_0^t R^{(\lambda_n)}(t, s) F(s, \|u_n\|_\infty v_n(s)) ds.$$

Note that, by (39) and the fact that  $F$  is a completely continuous mapping, for any  $\varepsilon > 0$ , there is  $m_\varepsilon \geq 0$  such that  $\|F(t, x) - F_\infty(t)x\| \leq \varepsilon\|x\| + m_\varepsilon$  for  $x \in E$ . Consequently, for each  $\varepsilon > 0$ , there exists  $n_\varepsilon \geq 1$  such that, for any  $n \geq n_\varepsilon$  and  $s \in [0, T]$ ,

$$(48) \quad \|\|u_n\|_\infty^{-1} F(s, \|u_n\|_\infty v_n(s)) - F_\infty(s)v_n(s)\| \leq \|u_n\|_\infty^{-1} (\varepsilon \|u_n\|_\infty \|v_n(s)\| + m_\varepsilon) \leq 2\varepsilon.$$

For each  $n \geq 1$ , put  $h_n(s) := \|u_n\|_\infty^{-1} F(s, \|u_n\|_\infty v_n(s))$  for  $s \in [0, T]$ . If  $s \in [0, T]$ , then using (48) and the compactness of  $F_\infty(s)$ , for arbitrary  $\varepsilon > 0$ , we deduce that

$$\beta(\{h_n(s)\}_{n \geq 1}) \leq \beta(F_\infty(s)(\{v_n(s)\}_{n \geq 1})) + 2\varepsilon = 2\varepsilon \quad \text{for } s \in [0, T],$$

which, by passing to the limit with  $\varepsilon \rightarrow 0$ , gives  $\beta(\{h_n(s)\}_{n \geq 1}) = 0$ . Obviously, we may also assume that  $\lambda_n \rightarrow \lambda_0$  for some  $\lambda_0 \in [0, 1]$ .

If  $\lambda_0 = 0$ , then note that

$$(49) \quad v_n(0) = R^{(\lambda_n)}(T, 0)^{k_n} v_n(0) + \left[ \lambda_n \sum_{k=0}^{k_n-1} R^{(\lambda_n)}(T, 0)^k \right] \left( \int_0^T R^{(\lambda_n)}(T, s) h_n(s) ds \right),$$

where  $(k_n)$  is an arbitrary sequence of positive integers. If we put  $k_n := [T/\lambda_n]$  for  $n \geq 1$ , then  $k_n \lambda_n \rightarrow T$  as  $n \rightarrow +\infty$  and, in view of Theorem 2.1 and Lemma 4.5, we get

$$\begin{aligned} \beta(\{v_n(0)\}_{n \geq 1}) &\leq e^{-\omega T} \beta(\{v_n(0)\}_{n \geq 1}) + \frac{1 - e^{-\omega T}}{\omega} \beta \left( \left\{ \int_0^T R^{(\lambda_n)}(T, s) h_n(s) ds \right\}_{n \geq 1} \right) \\ &\leq e^{-\omega T} \beta(\{v_n(0)\}_{n \geq 1}) + (1 - e^{-\omega T}) \omega^{-1} \int_0^T \beta(\{h_n(s)\}_{n \geq 1}) ds = e^{-\omega T} \beta(\{v_n(0)\}_{n \geq 1}). \end{aligned}$$

In consequence  $\beta(\{v_n(0)\}_{n \geq 1}) = 0$ . Furthermore, by (47) and Lemma 4.5, for any  $t \in [0, T]$ , one has

$$\begin{aligned} \beta(\{v_n(t)\}_{n \geq 1}) &\leq \beta(\{R^{(\lambda_n)}(t, 0)v_n(0)\}_{n \geq 1}) + \lambda_n \int_0^t \beta(\{R^{(\lambda_n)}(t, s)h_n(s)\}_{n \geq 1}) ds \\ &\leq \beta(\{v_n(0)\}_{n \geq 1}) + \int_0^t \beta(\{\lambda_n h_n(s)\}_{n \geq 1}) ds = 0, \end{aligned}$$

i.e.  $\beta(\{v_n(t)\}_{n \geq 1}) = 0$  for  $t \in [0, T]$ . Since, by (48), the set  $\{h_n\}_{n \geq 1}$  is bounded in  $C([0, T], E)$ , applying Proposition 3.1 (ii), we infer that  $\{v_n\}_{n \geq 1}$  is relatively compact in  $C([0, T], E)$  and without loss of generality, we assume that  $v_n \rightarrow v_0$  in  $C([0, T], E)$  and

$v_n(0) \rightarrow x_0 := v_0(0)$ . Furthermore, by (48), for arbitrary  $\varepsilon > 0$ , there exists  $n_\varepsilon \geq 1$  such that, for any  $n \geq n_\varepsilon$  and  $t \in [0, T]$ ,

$$\begin{aligned} \|v_n(t) - R^{(\lambda_n)}(t, 0)v_n(0)\| &\leq \lambda_n \int_0^t \|R^{(\lambda_n)}(t, s)F_\infty(s)v_n(s)\| ds + 2\lambda_n \varepsilon T \\ &\leq \lambda_n K \int_0^t \|v_n(s)\| ds + 2\lambda_n \varepsilon T \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

where  $K := \sup_{\tau \in [0, T]} \|F_\infty(\tau)\|$ . This together with Lemma 4.3 (i) imply that  $v_0(t) = x_0$  for any  $t \in E$  and in particular  $x_0 \neq 0$ , since  $\|v_0\|_\infty = 1$ . Hence, in view of (48) we find that

$$(50) \quad h_n(s) \rightarrow F_\infty(s)x_0 \quad \text{as } n \rightarrow \infty \text{ uniformly for } s \in [0, T].$$

Now fix an arbitrary  $\varepsilon > 0$  and take any sequence  $(k_n)$  of positive integers such that  $k_n \lambda_n \rightarrow \varepsilon$ . Applying Lemma 4.3 (ii), (iii) and (50) and passing to the limits in (49), we obtain

$$x_0 = S_{\hat{A}}(\varepsilon T)x_0 + \left[ \frac{1}{\varepsilon T} \int_0^{\varepsilon T} S_{\hat{A}}(\tau) d\tau \right] \left( \int_0^T F_\infty(s)x_0 ds \right),$$

i.e.

$$-\frac{1}{\varepsilon T} (S_{\hat{A}}(\varepsilon T)x_0 - x_0) = \frac{1}{\varepsilon T} \int_0^{\varepsilon T} S_{\hat{A}}(\tau) \hat{F}_\infty x_0 d\tau,$$

Hence, a passing to the limit with  $\varepsilon \rightarrow 0$  yields  $x_0 \in D(\hat{A})$  and  $(\hat{A} + \hat{F}_\infty)x_0 = 0$ , a contradiction proving (46) in the case  $\lambda_0 = 0$ .

If  $\lambda_0 \in (0, 1]$ , then, by (47) and Lemma 4.5,

$$\begin{aligned} \beta(\{v_n(0)\}_{n \geq 1}) &\leq e^{-\lambda_0 \omega T} \beta(\{v_n(0)\}_{n \geq 1}) + \lambda_0 \beta \left( \left\{ \int_0^T R^{(\lambda_n)}(T, s) h_n(s) ds \right\}_{n \geq 1} \right) \\ &\leq e^{-\lambda_0 \omega T} \beta(\{v_n(0)\}_{n \geq 1}) + \int_0^T \lambda_0 e^{-\lambda_0 \omega (T-s)} \beta(\{h_n(s)\}_{n \geq 1}) ds = e^{-\lambda_0 \omega T} \beta(\{v_n(0)\}_{n \geq 1}) \end{aligned}$$

and, consequently,  $\beta(\{v_n(0)\}_{n \geq 1}) = 0$ . Using again (47), for all  $t \in [0, T]$ ,

$$\beta(\{v_n(t)\}_{n \geq 1}) \leq e^{-\lambda_0 \omega t} \beta(\{v_n(0)\}_{n \geq 1}) + \int_0^t \lambda_0 e^{-\lambda_0 \omega (t-s)} \beta(\{h_n(s)\}_{n \geq 1}) ds = 0,$$

which gives  $\beta(\{v_n(t)\}_{n \geq 1}) = 0$  for any  $t \in [0, T]$ . Hence, due to the boundedness of  $\{h_n\}_{n \geq 1}$  in  $C([0, T], E)$  and Proposition 3.1 (ii), it follows that  $\{v_n\}_{n \geq 1}$  is relatively compact in  $C([0, T], E)$  and, without loss of generality, we assume that  $v_n \rightarrow v_0$  in  $C([0, T], E)$ . Then, using (47), (48) and Proposition 3.7, we infer, that for any  $t \in [0, T]$ ,

$$v_0(t) = R^{(\lambda_0)}(t, 0)v_0(0) + \lambda_0 \int_0^t R^{(\lambda_0)}(t, s)F_\infty(s)v_0(s) ds,$$

i.e.  $v_0$  is a nontrivial mild solution of (40) with  $\lambda = \lambda_0 \in (0, 1]$ , which is a contradiction proving (46).

Finally, (41) and (46) allow us to apply Theorem 4.10 to finish the proof.  $\square$



## 5 An application to hyperbolic partial differential equations

We end the paper with an example of a periodic problem for the hyperbolic evolution equation with a time-dependent damping term. Suppose that  $\Omega$  is an open bounded subset of  $\mathbb{R}^N$  and  $A : D(A) \rightarrow E$  is a positive self-adjoint linear operator with compact resolvents defined on a Hilbert space  $X := L^2(\Omega)$  with the scalar product and the corresponding norm denoted by  $(\cdot, \cdot)_0$  and  $|\cdot|_0$ , respectively. It is well known that such  $A$  determines its fractional power space  $X^{1/2}$  being a Hilbert space as well. If we denote the scalar product and the corresponding norm by  $(\cdot, \cdot)_{1/2}$  and  $|\cdot|_{1/2}$ , respectively, then it is known that

$$|u|_{1/2} \geq \lambda_1^{1/2} |u|_0 \quad \text{for any } u \in X^{1/2}$$

where  $\lambda_1 > 0$  is the smallest eigenvalue of  $A$ . Typical examples of  $A$  satisfying these conditions is  $-\Delta_D$ , where  $\Delta_D$  is the Laplacian operator with zero the Dirichlet boundary conditions or  $-\Delta_N + \alpha I$ , where  $\Delta_N$  is the Laplacian operator with the zero Neumann boundary conditions and  $\alpha > 0$ .

Consider a periodic problem

$$(51) \quad \begin{cases} u_{tt}(x, t) + \beta(t)u_t(x, t) + (Au)(x, t) + f(t, u(x, t)) = 0 & \text{in } \Omega \times (0, T] \\ u(x, 0) = u(x, T), \quad u_t(x, 0) = u_t(x, T) & \text{on } \partial\Omega, \end{cases}$$

where  $\beta : [0, T] \rightarrow \mathbb{R}$  is a  $T$ -periodic continuously differentiable function such that  $\beta(t) > 0$ , for  $t \in [0, T]$  and  $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function satisfying the following properties

$$(52) \quad \text{there is } L > 0 \text{ such that } |f(t, s_1) - f(t, s_2)| \leq L|s_1 - s_2| \text{ for } t \in [0, T], \quad s_1, s_2 \in \mathbb{R},$$

$$(53) \quad \text{there is } c > 0 \text{ such that } |f(t, s)| \leq c(1 + |s|) \text{ for } t \in [0, T], \quad s \in \mathbb{R},$$

$$(54) \quad f(0, s) = f(T, s) \text{ for } s \in \mathbb{R},$$

$$(55) \quad \lim_{|s| \rightarrow +\infty} \frac{f(t, s)}{s} = f_\infty \text{ uniformly with respect to } t \in [0, T],$$

for some  $f_\infty \in \mathbb{R} \setminus \sigma(A)$ . If we define  $N_f : [0, T] \times X \rightarrow X$  by  $N_f(t, u)(x) := f(t, u(x))$  for a.e.  $x \in \Omega$  and  $t \in [0, T]$ , then (51) can be rewritten as a system

$$\begin{cases} \dot{u}(t) = v(t) \\ \dot{v}(t) = -Au(t) - \beta(t)v(t) - N_f(t, u(t)), \end{cases} \quad \text{for } t \in [0, T]$$

and in a matrix form as

$$(56) \quad \dot{z}(t) = \mathbf{A}(t)z(t) + \mathbf{F}(t, z(t)), \quad \text{for } t \in [0, T]$$

with operators  $\mathbf{A}(t) : D(\mathbf{A}(t)) \rightarrow \mathbf{E}$ ,  $t \in [0, T]$ , on the separable Banach space  $\mathbf{E} := X^{1/2} \times X$ , defined by

$$(57) \quad D(\mathbf{A}(t)) := X^1 \times X^{1/2} \quad \text{for } t \in [0, T],$$

$$(58) \quad \mathbf{A}(t)(u, v) := (v, -Au - \beta(t)v) \quad \text{for } t \in [0, T], \quad (u, v) \in D(\mathbf{A}(t))$$

and  $\mathbf{F} : [0, T] \times \mathbf{E} \rightarrow \mathbf{E}$  given by  $\mathbf{F}(t, (u, v)) := (0, -N_f(t, u))$  for  $t \in [0, T]$ ,  $(u, v) \in \mathbf{E}$ .

We claim that the family  $\{\mathbf{A}(t)\}_{t \in [0, T]}$  and the map  $\mathbf{F}$  satisfy the assumptions of

Theorem 4.11 provided  $\mathbf{E}$  is endowed with a proper norm. To this end, for  $\eta > 0$ , define a new scalar product  $(\cdot, \cdot)_{\mathbf{E}, \eta} : \mathbf{E} \times \mathbf{E} \rightarrow \mathbb{R}$ , by

$$((u_1, v_1), (u_2, v_2))_{\mathbf{E}, \eta} := (u_1, u_2)_{1/2} + (v_1 + \eta u_1, v_2 + \eta u_2)_0.$$

Clearly it is a well-defined scalar product and the corresponding norm  $\|\cdot\|_{\mathbf{E}, \eta}$  is equivalent to the usual product norm  $\|\cdot\|$  in  $\mathbf{E} = X^{1/2} \times X$ . Let  $\beta_0 > 0$  be such that  $\beta(t) \geq \beta_0$  for  $t \in [0, T]$ . Putting  $\gamma := \max_{t \in [0, T]} \lambda_1^{-1/2}(\beta(t) + 1)$  for  $0 < \eta \leq 1$ , one has

$$\begin{aligned} (\mathbf{A}(t)(u, v), (u, v))_{\mathbf{E}, \eta} &= (v, u)_{1/2} + (-Au - \beta(t)v + \eta v, v + \eta u)_0 \\ &= (v, u)_{1/2} - (Au, v)_0 - (\beta(t)v, v)_0 + \eta|v|_0^2 - \eta(Au, u)_0 - \eta(\beta(t)v, u)_0 + \eta^2(v, u)_0 \\ &\leq -\eta|u|_{1/2}^2 - (\beta_0 - \eta)|v|_0^2 + \eta(\beta(t) + 1)|(v, u)_0 \\ &\leq -\eta|u|_{1/2}^2 - (\beta_0 - \eta)|v|_0^2 + \eta\gamma|u|_{1/2}|v| \\ &\leq -\eta|u|_{1/2}^2 - (\beta_0 - \eta)|v|_0^2 + (\eta/2)|u|_{1/2}^2 + (\eta\gamma^2/2)|v|_0^2 \\ &= -(\eta/2)|u|_{1/2}^2 - (\beta_0 - \eta - \eta\gamma^2/2)|v|_0^2, \end{aligned}$$

and therefore, decreasing  $\eta > 0$  if necessary, there exists  $\omega = \omega(\eta) > 0$  such that,  $(\mathbf{A}(t)(u, v), (u, v))_{\mathbf{E}, \eta} \leq -\omega\|(u, v)\|_{\mathbf{E}, \eta}^2$  for any  $(u, v) \in D(\mathbf{A}(t)) = X^1 \times X^{1/2}$ . Since  $\text{Im } \mathbf{A}(t) = \mathbf{E}$ , for  $t \in [0, T]$ , we infer that, for  $t \in [0, T]$ , the operator  $\mathbf{A}(t)$  is a generator of  $C_0$  semigroup satisfying

$$\|S_{\mathbf{A}(t)}(s)\|_{\mathbf{E}, \eta} \leq e^{-\omega s} \quad \text{for } s \geq 0$$

and, in particular, condition  $(Hyp'_1)$  holds. Moreover, observe that, for each  $(u, v) \in X^1 \times X^{1/2}$ , the map  $t \mapsto \mathbf{A}(t)(u, v) \in \mathbf{E}$  is continuously differentiable on  $[0, T]$  as  $\beta$  is so. Hence, in view of Proposition 3.8, the family  $\{\mathbf{A}(t)\}_{t \geq 0}$  satisfies also conditions  $(Hyp_2)$  and  $(Hyp_3)$  with  $\mathbf{V} := X^1 \times X^{1/2}$  equipped with the norm given by  $\|(u, v)\|_{\mathbf{V}} := \|\mathbf{A}(0)(u, v)\|_{\mathbf{E}, \eta} + \|(u, v)\|_{\mathbf{E}, \eta}$  for  $(u, v) \in \mathbf{V}$ . By the periodicity of  $\beta$ ,  $(Hyp_5)$  holds. Furthermore, observe that for  $(u, v) \in \mathbf{V} = X^1 \times X^{1/2}$

$$\mathbf{A}_0(u, v) := \frac{1}{T} \int_0^T \mathbf{A}(\tau)(u, v) d\tau = (v, -Au - \widehat{\beta}v)$$

where  $\widehat{\beta} := (1/T) \int_0^T \beta(\tau) d\tau$  and  $\text{Im } \mathbf{A}_0 = \mathbf{E}$  which implies  $(Hyp_4)$ . It can be easily verified that  $\mathbf{A}_0$  is closed and consequently,  $\widehat{\mathbf{A}} = \mathbf{A}_0$ . Since  $\beta$  is  $T$ -periodic function, we conclude that  $(Hyp_5)$  is also satisfied.

It may be checked that  $\mathbf{F}$  is continuous and, by (52), (53) and (53), satisfies conditions  $(F_1)$ ,  $(F_2)$  and  $(F_4)$ . Since the operator  $A$  has compact resolvents, the inclusion  $X^{1/2} \subset X$  is compact and therefore, both  $\mathbf{F}$  and  $\mathbf{F}_\infty : \mathbf{E} \rightarrow \mathbf{E}$ , given by  $\mathbf{F}_\infty(u, v) := (0, -f_\infty u)$ , are completely continuous. Furthermore observe that

$$(59) \quad \limsup_{\|(u, v)\| \rightarrow \infty, t \rightarrow t_0} \frac{\|\mathbf{F}(t, (u, v)) - \mathbf{F}_\infty(u, v)\|}{\|(u, v)\|} \leq \limsup_{|u|_{1/2} \rightarrow \infty, t \rightarrow t_0} \frac{|N_f(t, u) - f_\infty u|_0}{|u|_{1/2}}.$$

Now suppose that  $(u_n)$  is a sequence in  $X^{1/2}$  such that  $|u_n|_{1/2} \rightarrow +\infty$  as  $n \rightarrow +\infty$  and  $(t_n)$  in  $[0, T]$  is such that  $t_n \rightarrow t_0$  as  $n \rightarrow +\infty$ . If we put  $z_n := u_n/|u_n|_{1/2}$  for  $n \geq 1$ , then by the compactness of the inclusion  $X^{1/2} \subset X$ , there exists subsequence  $(z_{n_k})$  and  $z_0 \in X$  such that  $z_{n_k} \rightarrow z_0$  in  $X$  as  $k \rightarrow +\infty$ . Without lost of generality we may assume that

$z_{n_k}(x) \rightarrow z_0(x)$ , a.e. on  $\Omega$ , and there is  $g \in X$  such that, for each  $k \geq 1$ ,  $|z_{n_k}(x)| \leq g(x)$  a.e. on  $\Omega$ . Then, putting  $\mu_n := |u_n|_{1/2}$ , by (55), we find that

$$|\mu_{n_k}^{-1} f(t_{n_k}, x, \mu_{n_k} z_{n_k}(x)) - f_\infty z_{n_k}(x)|^2 \rightarrow 0 \quad \text{a.e. on } \Omega, \quad \text{as } k \rightarrow \infty$$

and, for each  $k \geq 1$ ,

$$|\mu_{n_k}^{-1} f(t_{n_k}, x, \mu_{n_k} z_{n_k}(x)) - f_\infty z_{n_k}(x)|^2 \leq g_0(x) \quad \text{a.e. on } \Omega$$

with  $g_0 := (c(m+g) + f_\infty g)^2$  where  $m := \sup\{\mu_n^{-1} \mid n \geq 1\}$ . Since  $\Omega$  is a bonded set,  $g_0$  is integrable and by the Lebesgue dominated convergence theorem

$$|N_f(t_{n_k}, u_{n_k}) - f_\infty u_{n_k}|_0 / |u_{n_k}|_{1/2} = \int_\Omega |\mu_{n_k}^{-1} f(t_{n_k}, x, \mu_{n_k} z_{n_k}(x)) - f_\infty z_{n_k}(x)|^2 dx \rightarrow 0$$

as  $k \rightarrow +\infty$ , which together with (59), implies that

$$\lim_{\|z\| \rightarrow \infty, t \rightarrow t_0} \frac{\|\mathbf{F}(t, z) - \mathbf{F}_\infty z\|}{\|z\|} = 0$$

and condition (39) is satisfied.

The following lemma will be helpful in verifying that (40) has no nontrivial  $T$ -periodic solutions for  $\lambda \in (0, 1)$ .

**Lemma 5.1** *Let  $A : D(A) \rightarrow E$  be a positive self-adjoint operator with compact resolvents on a Hilbert space  $X$  and, for fixed  $\bar{\beta} > 0$ ,  $\mathbf{A} : D(\mathbf{A}) \rightarrow \mathbf{E}$  be an linear operator on  $\mathbf{E} := X^{1/2} \times X^0$  given by  $D(\mathbf{A}) := X^1 \times X^{1/2}$  and*

$$\mathbf{A}(u, v) := (v, -Au - \bar{\beta}v) \quad \text{for } (u, v) \in D(\mathbf{A}).$$

*Let  $\mathbf{E}_k := X_k \times X_k$ ,  $k \geq 1$ , where  $X_k$  is the space spanned by the first  $k$  eigenvectors of  $A$  (corresponding to the first  $k$  smallest eigenvalues of  $A$ ), and  $\mathbf{A}_k : \mathbf{E}_k \rightarrow \mathbf{E}$  be given by*

$$\mathbf{A}_k(u, v) := \mathbf{A}(u, v) \quad \text{for } (u, v) \in \mathbf{E}_k.$$

*Then*

- (i)  $\mathbf{A}(\mathbf{E}_k) \subset \mathbf{E}_k$  for each  $k \geq 1$ ;
- (ii)  $R(\mu; \mathbf{A}_k)(u, v) = R(\mu; \mathbf{A})(u, v)$  for any  $k \geq 1$  and  $(u, v) \in \mathbf{E}_k$  and  $\mu > 0$ ;
- (iii)  $S_{\mathbf{A}_k}(t)(u, v) = S_{\mathbf{A}}(t)(u, v)$  for any  $k \geq 1$  and  $(u, v) \in \mathbf{E}_k$ .

**Proof.** (i) comes straightforwardly from the fact that  $A(X_k) \subset X_k$  for  $k \geq 1$ .

(ii) If  $(p, q) \in \mathbf{E}_k$  and  $(u, v) := R(\mu; \mathbf{A}_k)(p, q)$  for some  $\mu > 0$ , then  $\mu u - v = p$  and  $Au + (\mu + \bar{\beta})v = q$ , i.e.  $\mu(\mu + \bar{\beta})u + Au = (\mu + \bar{\beta})p + q \in X_k$ , which shows that  $u \in X_k$  and  $v = \mu u - p \in X_k$ . Therefore,  $\mu(u, v) - \mathbf{A}_k(u, v) = \mu(u, v) - \mathbf{A}(u, v) = (p, q)$ , that is  $R(\mu; \mathbf{A})(p, q) = (u, v) = R(\mu; \mathbf{A}_k)(p, q)$ .

(iii) follows from the Euler formula  $S_{\mathbf{A}}(t)(u, v) = \lim_{n \rightarrow +\infty} (n/t)^n R(n/t; \mathbf{A})^n(u, v)$ ,  $(u, v) \in \mathbf{E}_k$ , and (ii).  $\square$

**Lemma 5.2** *Let  $\{\mathbf{A}(t)\}_{t \in [0, T]}$  be given by (57), (58) and  $\{\mathbf{A}_k(t)\}_{t \in [0, T]}$ ,  $k \geq 1$ , be given by  $\mathbf{A}_k(t)(u, v) := \mathbf{A}(t)(u, v)$  for  $(u, v) \in \mathbf{E}_k$ . If  $\{\mathbf{R}(t, s)\}_{0 \leq s \leq t \leq T}$  and  $\{\mathbf{R}^{(k)}(t, s)\}_{0 \leq s \leq t \leq T}$ ,  $k \geq 1$  are the evolution systems determined by  $\{\mathbf{A}(t)\}_{t \in [0, T]}$  and  $\{\mathbf{A}_k(t) : \mathbf{E}_k \rightarrow \mathbf{E}_k\}_{t \in [0, T]}$ ,  $k \geq 1$ , respectively, then, for any  $k \geq 1$  and  $(u, v) \in \mathbf{E}_k$ ,*

$$\mathbf{R}^{(k)}(t, s)(u, v) = \mathbf{R}(t, s)(u, v) \quad \text{for } 0 \leq s \leq t \leq T.$$

**Proof.** By the construction of evolution systems (see [14, Ch. 5, Theorem 3.1]), for any  $(u, v) \in \mathbf{E}$ ,

$$(60) \quad \mathbf{R}(t, s)(u, v) = \lim_{n \rightarrow +\infty} \mathbf{R}_n(t, s)(u, v) \quad \text{for } 0 \leq s \leq t \leq T,$$

where  $\mathbf{R}_n(t, s) : \mathbf{E} \rightarrow \mathbf{E}$ ,  $n \geq 1$  are given by

$$\mathbf{R}_n(t, s) := \begin{cases} S_{\mathbf{A}(t_j^n)}(t-s) & \text{if } s, t \in [t_j^n, t_{j+1}^n], s \leq t, \\ S_{\mathbf{A}(t_r^n)}(t-t_r^n) \left( \prod_{j=l+1}^{r-1} S_{\mathbf{A}(t_j^n)}(T/n) \right) S_{\mathbf{A}(t_l^n)}(t_{l+1}^n - s) & \text{if } l < r \text{ and } s \in [t_l^n, t_{l+1}^n], \\ & t \in [t_r^n, t_{r+1}^n] \end{cases}$$

with  $t_j^n := (j/n)T$  for  $j = 0, 1, \dots, n$ . Similarly, for any  $k \geq 1$  and  $(u, v) \in \mathbf{E}_k$ ,

$$(61) \quad \mathbf{R}^{(k)}(t, s)(u, v) = \lim_{n \rightarrow +\infty} \mathbf{R}_n^{(k)}(t, s)(u, v) \quad \text{for } 0 \leq s \leq t \leq T.$$

where

$$\mathbf{R}_n^{(k)}(t, s) := \begin{cases} S_{\mathbf{A}_k(t_j^n)}(t-s) & \text{if } s, t \in [t_j^n, t_{j+1}^n], s \leq t, \\ S_{\mathbf{A}_k(t_r^n)}(t-t_r^n) \left( \prod_{j=l+1}^{r-1} S_{\mathbf{A}_k(t_j^n)}(T/n) \right) S_{\mathbf{A}_k(t_l^n)}(t_{l+1}^n - s) & \text{if } l < r \text{ and } s \in [t_l^n, t_{l+1}^n], \\ & t \in [t_r^n, t_{r+1}^n]. \end{cases}$$

Lemma 5.1 (iii) states that  $S_{\mathbf{A}(t_j^n)}(s)(u, v) = S_{\mathbf{A}_k(t_j^n)}(s)(u, v)$  for  $k \geq 1$ ,  $(u, v) \in \mathbf{E}_k$ ,  $s \geq 0$ ,  $n \geq 1$  and  $j \in \{1, \dots, n\}$ . Hence, using the formulae (60) and (61) completes the proof.  $\square$

**Lemma 5.3** (cf. [10], [16]) *If  $f : [0, T] \rightarrow X^0$  is continuous and  $(u, v) : [0, T] \rightarrow X^{1/2} \times X^0$  is a mild solution of*

$$(u(t), v(t))' = \mathbf{A}(t)(u(t), v(t)) + (0, f(t)), \quad t \in [0, T],$$

then

$$(62) \quad \frac{1}{2} \frac{d}{dt} |u(t)|_0^2 = (u(t), v(t))_0 \quad \text{for } t \in [0, T],$$

$$(63) \quad \frac{1}{2} \frac{d}{dt} \left( |u(t)|_{1/2}^2 + |v(t)|_0^2 \right) = -\beta(t)|v(t)|_0^2 + (f(t), v(t))_0 \quad \text{for } t \in [0, T].$$

**Proof.** Let  $(u, v) : [0, T] \rightarrow \mathbf{E}$  be a mild solution of

$$\dot{z}(t) = \mathbf{A}(t)z(t) + (0, f(t)), \quad t \in [0, T].$$

Put  $(\bar{u}_k, \bar{v}_k) := (\tilde{P}_k u(0), P_k v(0))$  for  $k \geq 1$ , where  $\tilde{P}_k : X^{1/2} \rightarrow X_k$  and  $P_k : X^0 \rightarrow X_k$  are the orthogonal projections. Furthermore, let  $(\tilde{u}_k, \tilde{v}_k) : [0, T] \rightarrow \mathbf{E}_k$  be the mild solution of

$$(64) \quad \begin{cases} (\dot{u}(t), \dot{v}(t)) = \mathbf{A}_k(t)(u(t), v(t)) + (0, P_k f(t)), & t \in [0, T] \\ (u(0), v(0)) = (\bar{u}_k, \bar{v}_k) \end{cases}$$

and for each  $k \geq 1$  define  $(u_k, v_k) : [0, T] \rightarrow \mathbf{E}$  by  $(u_k(t), v_k(t)) := (\tilde{u}_k(t), \tilde{v}_k(t))$  for  $t \in [0, T]$ . Then

$$(\tilde{u}_k(t), \tilde{v}_k(t)) = \mathbf{R}^{(k)}(t, 0)(\bar{u}_k, \bar{v}_k) + \int_0^t \mathbf{R}^{(k)}(t, s)(0, P_k f(s)) ds \quad \text{for } t \in [0, T],$$

and by Lemma 5.2, one gets

$$(u_k(t), v_k(t)) = \mathbf{R}(t, 0)(\bar{u}_k, \bar{v}_k) + \int_0^t \mathbf{R}(t, s)(0, P_k f(s)) ds \quad \text{for } t \in [0, T].$$

Further, since  $(\bar{u}_k, \bar{v}_k) \rightarrow (u(0), v(0))$  in  $\mathbf{E}$  and  $P_k f(t) \rightarrow f(t)$  in  $X^0$  as  $k \rightarrow +\infty$  uniformly for  $t \in [0, T]$ , by Proposition 3.1 (i) we infer that  $(u_k, v_k) \rightarrow (u, v)$  in  $C([0, T], \mathbf{E})$ . Finally, treating (64) as a system of ordinary differential equations with the family  $\{\mathbf{A}_k(t)\}_{t \in [0, T]}$  of bounded operators we see that  $(\tilde{u}_k, \tilde{v}_k)$  is, in particular, a classical solution. Therefore, we obtain

$$\begin{aligned} (u_k(t), \dot{u}_k(t))_{1/2} &= (u_k(t), v_k(t))_{1/2} \\ (v_k(t), \dot{v}_k(t))_0 &= -(u_k(t), v_k(t))_{1/2} - \beta(t)|v_k(t)|_0^2 + (v_k(t), P_k f(t))_0, \end{aligned}$$

for any  $t \in [0, T]$  and, as a result,

$$\frac{1}{2} \frac{d}{dt} (|u_k(t)|_{1/2}^2 + |v_k(t)|_0^2) = -\beta(t)|v_k(t)|_0^2 + (v_k(t), P_k f(t))_0 \quad \text{for } t \in [0, T].$$

Thus, we see that both, the functions  $|u_k|_{1/2}^2 + |v_k|_0^2$ ,  $k \geq 1$  and their derivatives converge uniformly on  $[0, T]$ , which gives (63). To see (62) observe that

$$(65) \quad \frac{1}{2} \frac{d}{dt} |u_k(t)|_0^2 = (u_k(t), \dot{u}_k(t))_0 = (u_k(t), v_k(t))_0 \quad \text{for } t \in [0, T]$$

and  $u_k(t) \rightarrow u(t)$ ,  $v_k(t) \rightarrow v(t)$  is a space  $X^0$ , as  $k \rightarrow \infty$ , uniformly with respect to  $t \in [0, T]$ . Hence we see that the functions  $|u_k|_0^2$ ,  $k \geq 1$ , and their derivatives are convergent uniformly and, by (65), we are done.  $\square$

Now return to our considerations of (56) and suppose that, for some,  $\lambda \in (0, 1]$ ,  $(u, v) : [0, T] \rightarrow E$  is a  $T$ -periodic mild solution of

$$(u, v)' = \lambda(\mathbf{A}(t)(u(t), v(t)) + \mathbf{F}_\infty(u(t), v(t))), \quad t \in [0, T].$$

If view of Lemma 5.3 we get

$$\frac{1}{2} \frac{d}{dt} (|u(t)|_{1/2}^2 + |v(t)|_0^2) = -\lambda\beta(t)|v(t)|_0^2 - \lambda f_\infty(u(t), v(t))_0$$

and, after integrating and using (62), one has

$$0 < \int_0^T \lambda\beta(t)|v(t)|_0^2 dt = -\frac{1}{2} \int_0^T \lambda f_\infty(|u(t)|_0^2)' dt = 0,$$

a contradiction proving that (40) has no nontrivial  $T$ -periodic solutions.

Finally, by a direct calculation, we see that  $\text{Ker}(\hat{\mathbf{A}} + \mathbf{F}_\infty) = \{0\}$  since  $f_\infty \notin \sigma(A)$ . Thus, in view of Theorem 4.11, problem (51) admits a  $T$ -periodic solution in the sense that (56) has a  $T$ -periodic mild solution.

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